Semi-Substructural Logics à la Lambek with Symmetry

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Abstract

This work studies the proof theory and ternary relational semantics of left (right) skew monoidal closed categories and skew monoidal bi-closed categories—both symmetric and non-symmetric—from the perspective of non-associative Lambek calculus. Uustalu et al. used sequents with stoup (the leftmost position of an antecedent that can be either empty or a single formula) to deductively model left skew monoidal closed categories, yielding results regarding proof identities and categorical coherence. However, their syntax does not work well when modeling right skew monoidal closed and skew monoidal bi-closed categories, whether symmetric or non-symmetric.

We solve the problem via more flexible and equivalent frameworks to characterize the categories above: tree sequent calculus (where antecedents are binary trees) and axiomatic calculus (where antecedents are a single formula), inspired by works on non-associative Lambek calculus. Moreover, we prove that the axiomatic calculi are sound and complete with respect to their ternary relational models. We also prove a correspondence between frame conditions and structural laws, providing an algebraic way to understand the relationship between the left and right skew monoidal closed categories, encompassing both symmetric and non-symmetric variants.

1 Introduction

Substructural logics are logic systems that lack at least one of the structural rules, weakening, contraction, and exchange. Joachim Lambek's syntactic calculus [18] is a well-known example that disallows weakening, contraction, and exchange. Another example, linear logic, proposed by Jean-Yves Girard [14], is a substructural logic in which weakening and contraction are in general disallowed but can be recovered for some formulae via modalities. Substructural logics have been found in numerous applications from computational analysis of natural languages to the development of resource-sensitive programming languages.

Left skew monoidal categories [25] are a weaker variant of MacLane's monoidal categories where the structural morphisms of associativity and unitality are not required to be bidirectional, they are natural transformations with a particular orientation. Therefore, they can be seen as *semi-associative* and *semi-unital* variants of monoidal categories. Left skew monoidal categories arise naturally in the semantics of programming languages [2], while the concept of semi-associativity is connected with combinatorial structures like the Tamari lattice and Stasheff associahedra [36, 21].

In recent years, Tarmo Uustalu, Niccolò Veltri, and Noam Zeilberger started a research project on *semi-substructural* logics, which is inspired by a series of developments on left skew monoidal categories and related variants by Szlachányi, Street, Bourke, Lack and others [25, 16, 24, 17, 8, 5, 6, 7].

We call the languages of left skew monoidal categories and their variants *semi-substructural* logics, because they are intermediate logics between (certain fragments of) non-associative and associative intuitionistic linear logic (or Lambek calculus). Semi-associativity and semi-unitality are encoded as follows. Sequents are in the form $S \mid \Gamma \vdash A$, where the antecedent consists of an optional formula S, called stoup, adapted from Girard [15], and an ordered list of formulae Γ. The succedent is a single formula A. We restrict the application of introduction rules in an appropriate way to allow only one of the directions of associativity and unitality.

This approach has successfully captured languages for a variety of categories, including (*i*) left skew semigroup [36], (*ii*) left skew monoidal [30], (*iii*) left skew (prounital) closed [28], (*iv*) left skew monoidal closed categories [26, 32], and (*v*) left distributive skew monoidal categories with finite products and coproducts [33] through skew variants of the fragments of non-commutative intuitionistic linear logic consisting of combinations of connectives $(\mathsf{I}, \otimes, \multimap, \wedge, \vee)$. Additionally, discussions have covered partial normality conditions, in which one or more structural morphisms are allowed to have an inverse [29], as well as extensions with skew exchange à la Bourke and Lack [31, 33, 34].

In all of the aforementioned works, internal languages of left skew monoidal categories and their variants are characterized in a similar way which we call sequent calculus à la Girard. These calculi with sequents of the form $S \mid \Gamma \vdash A$ are cut-free and by their rule design, they are decidable. Moreover, they all admit sound and complete subcalculi inspired by Andreoli's focusing [3] in which rules are restricted to be applied in a specific order. A focused calculus provides an algorithm to solve both the proof identity problems for its non-focused calculus and coherence problems for its corresponding variant of left skew monoidal category.

By reversing all structural morphisms and modifying coherence conditions in left skew monoidal closed categories, right skew monoidal closed categories emerge [27]. Moreover, skew monoidal bi-closed categories are defined by appropriately integrating left and right skew monoidal closed structures. It is natural for us to consider sound sequent calculi for these categories. However, the implication rules are not well-behaved when just modeling right skew monoidal closed categories with sequent calculus à la Girard.

The problem stems from the skew structure concealed within the flat antecedent of $S \mid \Gamma \vdash A$. While the antecedent $S \mid \Gamma$ is defined similarly to an ordered list, it is actually a tree associating to the left. We start in Section 2, by introducing the sequent calculus à la Girard (`LSkG`) for left skew monoidal closed categories from [26] and its equivalent sequent calculus à la Lambek (`LSkT`), which is inspired by sequent calculus for non-associative Lambek calculus [9, 22] with trees as antecedents.

Associative (non-associative) Lambek calculus can be extended with permutation by adding a rule of exchange [22]. In the commutative version of the Lambek calculus, two implications $\diagdown$ and $\diagup$ collapse into one, i.e. for any formulae A and B, $A \diagdown B$ is logically equivalent to $B \diagup A$. This leads to a natural question to consider semi-substructural logics with permutation. An immediate idea is adding an axiom of permutation directly into the calculus. However, the axiom ex makes the calculus fully normal, i.e. α^{-1}, λ^{-1}, and ρ^{-1} are provable. Veltri addressed the addition of permutation to sequent calculi for symmetric skew monoidal and skew closed categories [31, 34]. Here, we extend this work by generalizing these results to symmetric right skew categories and symmetric skew monoidal bi-closed categories.

In Section 3, we introduce definitions of left (right) skew monoidal closed categories and skew monoidal bi-closed categories, and normality conditions for skew categories. In Section 4, we describe two calculi that characterize skew monoidal bi-closed categories: one is an axiomatic calculus (`SkMBiCA`), while the other is a sequent calculus (`SkMBiCT`) similar to the multimodal non-associative Lambek calculus [20]. In Section 5, we introduce the relational semantics for `SkMBiCA` via preordered sets of possible worlds with ternary relations. Furthermore, we show a correspondence theorem (Theorem 5.7) between conditions on ternary relations and structural laws on any frame. The theorem allows us to prove a thin version of main theorems in [27]. Finally, in Section 6, we incorporate commutativity into semi-substructural logics from both syntactic and semantic perspective following the method in [31, 34] and extend the result to symmetric right skew categories and symmetric skew monoidal bi-closed categories.

Publication History This paper is an extended version of [35]. Compared to the conference version, we have added Lemmata 2.10 and 4.3, which are essential to the proof of equivalence of calculi (`LSkG` and `LSkT` for the former and `SkMBiCA` and `SkMBiCT` for the latter) and detailed the proof of Theorem 4.5. The whole Section 6, studying the syntax and semantics of semi-substructural logics with permutation, is new.

2 Sequent Calculus

We recall the sequent calculus à la Girard for left skew monoidal closed categories from [26], which is a skew variant of non-commutative multiplicative intuitionistic linear logic.

Formulae (Fma) in `LSkG` are inductively generated by the grammar $A, B ::= X \mid \mathsf{I} \mid A \otimes B \mid A \multimap B$, where X comes from a set At of atoms, I is a multiplicative unit, $\otimes$ is multiplicative conjunction and $\multimap$ is a linear implication.
A sequent is a triple of the form $S \mid \Gamma \vdash_{\mathsf{G}} A$, where the antecedent splits into: an optional formula S, called *stoup* [15], and an ordered list of formulae Γ and succedent A is a single formula. The symbol S consistently denotes a stoup, meaning S can either be a single formula or empty, indicated as $S = -$; furthermore, X, Y, and Z always represent atomic formulae.

Definition 2.1. *Derivations in* LSkG *are generated recursively by the following rules:*

$$\frac{}{A \mid \ \vdash_{\mathsf{G}} A}\ \mathsf{ax} \qquad \frac{- \mid \Gamma \vdash_{\mathsf{G}} A \quad B \mid \Delta \vdash_{\mathsf{G}} C}{A \multimap B \mid \Gamma, \Delta \vdash_{\mathsf{G}} C}\ {\multimap}\mathsf{L} \qquad \frac{- \mid \Gamma \vdash_{\mathsf{G}} C}{\mathsf{I} \mid \Gamma \vdash_{\mathsf{G}} C}\ \mathsf{IL}$$

$$\frac{A \mid B, \Gamma \vdash_{\mathsf{G}} C}{A \otimes B \mid \Gamma \vdash_{\mathsf{G}} C}\ \otimes\mathsf{L} \qquad \frac{A \mid \Gamma \vdash_{\mathsf{G}} C}{- \mid A, \Gamma \vdash_{\mathsf{G}} C}\ \mathsf{pass} \qquad \frac{S \mid \Gamma, A \vdash_{\mathsf{G}} B}{S \mid \Gamma \vdash_{\mathsf{G}} A \multimap B}\ {\multimap}\mathsf{R}$$

$$\frac{}{- \mid \ \vdash_{\mathsf{G}} \mathsf{I}}\ \mathsf{IR} \qquad \frac{S \mid \Gamma \vdash_{\mathsf{G}} A \quad - \mid \Delta \vdash_{\mathsf{G}} B}{S \mid \Gamma, \Delta \vdash_{\mathsf{G}} A \otimes B}\ \otimes\mathsf{R}$$

The inference rules of LSkG are similar to the ones in the sequent calculus for non-commutative multiplicative intuitionistic linear logic (NMILL) [1], but with some crucial differences:

1. The left logical rules IL, $\otimes$L and $\multimap$L, read bottom-up, are only allowed to be applied on the formula in the stoup position.

2. The right tensor rule $\otimes$R, read bottom-up, splits the antecedent of a sequent $S \mid \Gamma, \Delta \vdash_{\mathsf{G}} A \otimes B$ and in the case where S is a formula, S is always moved to the stoup of the left premise, even if Γ is empty.

3. The presence of the stoup distinguishes two types of antecedents, $A \mid \Gamma$ and $- \mid A, \Gamma$. The structural rule pass (for 'passivation'), read bottom-up, allows the moving of the leftmost formula in the context to the stoup position whenever the stoup is empty.

4. The logical connectives of NMILL (and associative Lambek calculus) typically include two ordered implications $\setminus$ and $/$, which are two variants of linear implication arising from the removal of the exchange rule from intuitionistic linear logic. In LSkG, only the right residuation ($B / A = A \multimap B$) of Lambek calculus is present.

For a more detailed explanation and a linear logical interpretation of LSkG, see [26, Section 2].

Theorem 2.2. LSkG *is cut-free, i.e. the rules*

$$\frac{\overset{f}{S \mid \Gamma \vdash_{\mathsf{G}} A} \quad \overset{g}{A \mid \Delta \vdash_{\mathsf{G}} C}}{S \mid \Gamma, \Delta \vdash_{\mathsf{G}} C}\ \mathsf{scut} \qquad \frac{\overset{f}{- \mid \Gamma \vdash_{\mathsf{G}} A} \quad \overset{g}{S \mid \Delta_0, A, \Delta_1 \vdash_{\mathsf{G}} C}}{S \mid \Delta_0, \Gamma, \Delta_1 \vdash_{\mathsf{G}} C}\ \mathsf{ccut}$$

are admissible.

Proof. The proof proceeds by induction on the height of derivations and the complexity of cut formulae. Specifically, for scut, we first perform induction on the left premise f, and if necessary, we perform subinduction on g or the complexity of the cut formula A. For ccut, we start by performing induction on the right premise g instead. The cases other than $\multimap$L and $\multimap$R have been discussed in [30, Lemma 5], so we will only elaborate on the cases of $\multimap$.

We first deal with scut. If $f = {\multimap}\mathsf{L}(f', f'')$, then we permute scut up, i.e.

$$\frac{\dfrac{\overset{f'}{- \mid \Gamma \vdash_{\mathsf{G}} A'} \quad \overset{f''}{B' \mid \Delta \vdash_{\mathsf{G}} A}}{A' \multimap B' \mid \Gamma, \Delta \vdash_{\mathsf{G}} A}\,{\multimap}\mathsf{L} \quad \overset{g}{A \mid \Lambda \vdash_{\mathsf{G}} C}}{A' \multimap B' \mid \Gamma, \Delta, \Lambda \vdash_{\mathsf{G}} C}\,\mathsf{scut}$$

$$\mapsto \quad \frac{\overset{f'}{- \mid \Gamma \vdash_{\mathsf{G}} A'} \quad \dfrac{\overset{f''}{B' \mid \Delta \vdash_{\mathsf{G}} A} \quad \overset{g}{A \mid \Lambda \vdash_{\mathsf{G}} C}}{B' \mid \Delta, \Lambda \vdash_{\mathsf{G}} C}\,\mathsf{scut}}{A' \multimap B' \mid \Gamma, \Delta, \Lambda \vdash_{\mathsf{G}} C}\,{\multimap}\mathsf{L}$$

If $f = {\multimap}\mathsf{R}\ f'$, then we perform a subinduction on g:

- If $g = {\multimap}\mathsf{L}(g', g'')$, then

$$\frac{\dfrac{\overset{f'}{S \mid \Gamma, A \vdash_{\mathsf{G}} B}}{S \mid \Gamma \vdash_{\mathsf{G}} A \multimap B}\,{\multimap}\mathsf{R} \quad \dfrac{\overset{g'}{- \mid \Delta \vdash_{\mathsf{G}} A} \quad \overset{g''}{B \mid \Lambda \vdash_{\mathsf{G}} C}}{A \multimap B \mid \Delta, \Lambda \vdash_{\mathsf{G}} C}\,{\multimap}\mathsf{L}}{S \mid \Gamma, \Delta, \Lambda \vdash_{\mathsf{G}} C}\,\mathsf{scut}$$

$$\mapsto \quad \frac{\overset{g'}{- \mid \Delta \vdash_{\mathsf{G}} A} \quad \dfrac{\overset{f'}{S \mid \Gamma, A \vdash_{\mathsf{G}} B} \quad \overset{g''}{B \mid \Lambda \vdash_{\mathsf{G}} C}}{S \mid \Gamma, A, \Lambda \vdash_{\mathsf{G}} C}\,\mathsf{scut}}{S \mid \Gamma, \Delta, \Lambda \vdash_{\mathsf{G}} C}\,\mathsf{ccut}$$

  where the complexity of the cut formulae is reduced.

- For other rules, we permute scut up. For example, if $g = {\multimap}\mathsf{R}\ g'$, then

$$\frac{\dfrac{\overset{f'}{S \mid \Gamma, A \vdash_{\mathsf{G}} B}}{S \mid \Gamma \vdash_{\mathsf{G}} A \multimap B}\,{\multimap}\mathsf{R} \quad \dfrac{\overset{g'}{A \multimap B \mid \Delta, A' \vdash_{\mathsf{G}} B'}}{A \multimap B \mid \Delta \vdash_{\mathsf{G}} A' \multimap B'}\,{\multimap}\mathsf{R}}{S \mid \Gamma, \Delta \vdash_{\mathsf{G}} A' \multimap B'}\,\mathsf{scut}$$

$$\mapsto \quad \dfrac{\dfrac{\dfrac{\overset{f'}{S \mid \Gamma, A \vdash_{\mathsf{G}} B}}{S \mid \Gamma \vdash_{\mathsf{G}} A \multimap B}\,{\multimap}\mathsf{R} \quad \overset{g'}{A \multimap B \mid \Delta, A' \vdash_{\mathsf{G}} B'}}{S \mid \Gamma, \Delta, A' \vdash_{\mathsf{G}} B'}\,\mathsf{scut}}{S \mid \Gamma, \Delta \vdash_{\mathsf{G}} A' \multimap B'}\,{\multimap}\mathsf{R}$$

For ccut, if $g = {\multimap}\mathsf{R}\ g'$, then we permute ccut up. If $g = {\multimap}\mathsf{L}(g', g'')$, we permute ccut up as well, but depending on where the cut formula is placed, we either apply ccut on f and g' or f and g''. □

Moreover, LSkG is sound and complete wrt. left skew monoidal closed categories [26, Theorem 3.2].

By soundness and completeness, similar to the result in [30] for skew monoidal categories, we mean that LSkG is deductively equivalent to the axiomatic characterization of the free left skew monoidal closed category.

Definition 2.3. *Derivations in the axiomatic calculus of left skew monoidal closed category are generated by the following rules.*

$$\frac{}{A \vdash_{\mathsf{L}} A}\ \mathsf{id} \qquad \frac{A \vdash_{\mathsf{L}} B \quad B \vdash_{\mathsf{L}} C}{A \vdash_{\mathsf{L}} C}\ \mathsf{comp} \qquad \frac{A \vdash_{\mathsf{L}} C \quad B \vdash_{\mathsf{L}} D}{A \otimes B \vdash_{\mathsf{L}} C \otimes D}\ \otimes$$

$$\frac{C \vdash_{\mathsf{L}} A \quad B \vdash_{\mathsf{L}} D}{A \multimap B \vdash_{\mathsf{L}} C \multimap D}\ \multimap \qquad \frac{}{\mathsf{I} \otimes A \vdash_{\mathsf{L}} A}\ \lambda \qquad \frac{}{A \vdash_{\mathsf{L}} A \otimes \mathsf{I}}\ \rho$$

$$\frac{}{(A \otimes B) \otimes C \vdash_{\mathsf{L}} A \otimes (B \otimes C)}\ \alpha \qquad \frac{A \otimes B \vdash_{\mathsf{L}} C}{A \vdash_{\mathsf{L}} B \multimap C}\ \pi$$

In particular, this is a semi-unital and semi-associative variation of Moortgat and Oehrle's calculus [22, Chapter 4] of non-associative Lambek calculus (NL), where only right residuation is present. We only care about sequent derivability in this section, therefore we omit the congruence relations on sets of derivations $A \vdash_{\mathsf{L}} B$ and $S \mid \Gamma \vdash_{\mathsf{G}} A$ that identify certain pairs of derivations. However, the congruence relations are essential for these calculi being correct characterizations of the free left skew monoidal closed category.

The calculus LSkG, being an equivalent presentation of a skew version of NL, provides an effective procedure to determine formulae derivability in LSkNL. In other words, for any formula A, $\vdash_{\mathsf{L}} A$ if and only if $- \mid\ \vdash_{\mathsf{G}} A$. Exhaustive proof search in LSkG always terminates, so for any A, either it finds a proof or it fails and there is no proof

Adapted from [22], we define trees inductively by the grammar $T ::= \mathsf{Fma} \mid - \mid (T, T)$, where $-$ is an empty tree. A context is a tree with a hole defined recursively as $\mathcal{C} ::= [\cdot] \mid (\mathcal{C}, T) \mid (T, \mathcal{C})$. The substitution of a tree into a hole is defined recursively:

$$\begin{array}{rcl} subst([\cdot], U) & = & U \\ subst((T', \mathcal{C}), U) & = & (T', subst(\mathcal{C}, U)) \\ subst((\mathcal{C}, T'), U) & = & (subst(\mathcal{C}, U), T') \end{array}$$

We use $T[\cdot]$ to denote a context and $T[U]$ to abbreviate $subst(T[\cdot], U)$. Sometimes we omit parentheses for trees when it does not cause ambiguity. Sequents in LSkT are in the form $T \vdash_{\mathsf{T}} A$ where T is a tree and A is a single formula.

Definition 2.4. *Derivations in* LSkT *are generated recursively by following rules:*

$$\frac{}{A \vdash_{\mathsf{T}} A}\ \mathsf{ax}$$

$$\frac{T[-] \vdash_{\mathsf{T}} C}{T[\mathsf{I}] \vdash_{\mathsf{T}} C}\ \mathsf{IL} \qquad \frac{}{- \vdash_{\mathsf{T}} \mathsf{I}}\ \mathsf{IR} \qquad \frac{T[A, B] \vdash_{\mathsf{T}} C}{T[A \otimes B] \vdash_{\mathsf{T}} C}\ \otimes\mathsf{L} \qquad \frac{T \vdash_{\mathsf{T}} A \quad U \vdash_{\mathsf{T}} B}{T, U \vdash_{\mathsf{T}} A \otimes B}\ \otimes\mathsf{R}$$

$$\frac{U \vdash_{\mathsf{T}} A \quad T[B] \vdash_{\mathsf{T}} C}{T[A \multimap B, U] \vdash_{\mathsf{T}} C}\ \multimap\mathsf{L} \qquad \frac{T, A \vdash_{\mathsf{T}} B}{T \vdash_{\mathsf{T}} A \multimap B}\ \multimap\mathsf{R}$$

$$\frac{T[U_0, (U_1, U_2)] \vdash_{\mathsf{T}} C}{T[(U_0, U_1), U_2] \vdash_{\mathsf{T}} C}\ \mathsf{assoc} \qquad \frac{T[U] \vdash_{\mathsf{T}} C}{T[-, U] \vdash_{\mathsf{T}} C}\ \mathsf{unitL} \qquad \frac{T[U, -] \vdash_{\mathsf{T}} C}{T[U] \vdash_{\mathsf{T}} C}\ \mathsf{unitR}$$

This calculus is similar to the ones for NL [22] and NL with unit [9] but with semi-associative (assoc) and semi-unital (unitL and unitR) rules. The structural rule unitL, read bottom-up, removes an empty tree from the left. It helps us to correctly characterize the axiom λ in LSkT, i.e. $\mathsf{I} \otimes A \vdash_{\mathsf{T}} A$ is derivable while $A \vdash_{\mathsf{T}} \mathsf{I} \otimes A$ is not. Analogously for the rule unitR, from a bottom-up perspective, adds an empty tree from the right, and we cannot capture ρ in LSkT without unitR (a double question mark ?? means that no rule can be applied to close the derivation):

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash_{\mathsf{T}} A}\ \mathsf{ax}}{-, A \vdash_{\mathsf{T}} A}\ \mathsf{unitL}}{\mathsf{I}, A \vdash_{\mathsf{T}} A}\ \mathsf{IL}}{\mathsf{I} \otimes A \vdash_{\mathsf{T}} A}\ \otimes\mathsf{L} \qquad \dfrac{\dfrac{\dfrac{??}{X \vdash_{\mathsf{T}} \mathsf{I}} \quad \dfrac{??}{- \vdash_{\mathsf{T}} X}}{X, - \vdash_{\mathsf{T}} \mathsf{I} \otimes X}\ \otimes\mathsf{R}}{X \vdash_{\mathsf{T}} \mathsf{I} \otimes X}\ \mathsf{unitR}$$

$$\dfrac{\dfrac{\dfrac{}{A \vdash_{\mathsf{T}} A}\ \mathsf{ax} \quad \dfrac{}{- \vdash_{\mathsf{T}} \mathsf{I}}\ \mathsf{IR}}{A, - \vdash_{\mathsf{T}} A \otimes \mathsf{I}}\ \otimes\mathsf{R}}{A \vdash_{\mathsf{T}} A \otimes \mathsf{I}}\ \mathsf{unitR} \qquad \dfrac{\dfrac{\dfrac{??}{X, - \vdash_{\mathsf{T}} X}}{X, \mathsf{I} \vdash_{\mathsf{T}} X}\ \mathsf{IL}}{X \otimes \mathsf{I} \vdash_{\mathsf{T}} X}\ \otimes\mathsf{L}$$

Theorem 2.5. LSkT *is cut-free, i.e. the rule*

$$\dfrac{\overset{f}{U \vdash_{\mathsf{T}} A} \quad \overset{g}{T[A] \vdash_{\mathsf{T}} C}}{T[U] \vdash_{\mathsf{T}} C}\ \mathsf{cut}$$

is admissible.

Proof. We perform induction on the structure of derivation f of the left premise, and if necessary, we perform subinduction on the derivation g or the complexity of the cut formula A. Cases of logical rules $\mathsf{ax}, \otimes\mathsf{L}, \otimes\mathsf{R}, \multimap\mathsf{L}$, and $\multimap\mathsf{R}$ have been discussed in [22], so we only elaborate on the new cases arising in LSkT.

- The first new case is that $f = \mathsf{IR}$, then we inspect the structure of g.
  - If $g = \mathsf{ax} : \mathsf{I} \vdash_{\mathsf{T}} \mathsf{I}$, then we define $\mathsf{cut}(\mathsf{IR}, \mathsf{ax}) = \mathsf{IR}$.
  - If $g = \mathsf{IL}\ g'$, then there are two subcases:
    - ∗ if the I introduced by IL is the cut formula, then we define

$$\dfrac{\dfrac{}{- \vdash_{\mathsf{T}} \mathsf{I}}\ \mathsf{IR} \quad \dfrac{\overset{g'}{T[-] \vdash_{\mathsf{T}} C}}{T[\mathsf{I}] \vdash_{\mathsf{T}} C}\ \mathsf{IL}}{T[-] \vdash_{\mathsf{T}} C}\ \mathsf{cut} \quad \mapsto \quad \overset{g'}{T[-] \vdash_{\mathsf{T}} C}$$

    - ∗ if the I introduced by IL is not the cut formula, then we define

$$\dfrac{\dfrac{}{- \vdash_{\mathsf{T}} \mathsf{I}}\ \mathsf{IR} \quad \dfrac{\overset{g'}{T[-] \vdash_{\mathsf{T}} C}}{T[\mathsf{I}] \vdash_{\mathsf{T}} C}\ \mathsf{IL}}{T^{\{\mathsf{I}:=-\}}[\mathsf{I}] \vdash_{\mathsf{T}} C}\ \mathsf{cut}$$

$$\mapsto \quad \dfrac{\dfrac{\dfrac{}{- \vdash_{\mathsf{T}} \mathsf{I}}\ \mathsf{ax} \quad \overset{g'}{T[-] \vdash_{\mathsf{T}} C}}{T^{\{\mathsf{I}:=-\}}[-] \vdash_{\mathsf{T}} C}\ \mathsf{cut}}{T^{\{\mathsf{I}:=-\}}[\mathsf{I}] \vdash_{\mathsf{T}} C}\ \mathsf{IL}$$

      where $T^{\{\mathsf{I}:=-\}}[\cdot]$ means that a formula occurrence I at some fixed position in the context has been replaced by $-$.

  - If $g = \mathcal{R}\ g'$, where $\mathcal{R}$ is a one-premise rule other than IL, then $\mathsf{cut}(\mathsf{IR}, \mathcal{R}\ g') = \mathcal{R}(\mathsf{cut}(\mathsf{IR}, g'))$.
  - The cases of an arbitrary two-premises rule are similar.

- The only other new cases are IL and the structural rules, which are all one-premise left rules, where we can permute cut upwards. For example, if $f = \mathsf{unitL}\ f'$, then we define

$$\dfrac{\dfrac{\overset{f'}{T'[U] \vdash_{\mathsf{T}} A}}{T'[-,U] \vdash_{\mathsf{T}} A}\ \mathsf{unitL} \quad \overset{g}{T[A] \vdash_{\mathsf{T}} C}}{T[T'[-,U]] \vdash_{\mathsf{T}} C}\ \mathsf{cut}$$

$$\mapsto \quad \dfrac{\dfrac{\overset{f'}{T'[U] \vdash_{\mathsf{T}} A} \quad \overset{g}{T[A] \vdash_{\mathsf{T}}}}{T[T'[U]] \vdash_{\mathsf{T}} C}\ \mathsf{cut}}{T[T'[-,U]] \vdash_{\mathsf{T}} C}\ \mathsf{unitL}$$

The other cases are similar.

□

The proof of equivalence relies on the following lemmata and definitions.

Definition 2.6. *For any tree T, T^* is the formula obtained from T by replacing commas with $\otimes$ and $-$ with I, respectively.*

Lemma 2.7. *For any context $T[\cdot]$ and tree U, $T[U]^* = T[U^*]^*$.*

Proof. The proof proceeds by induction on the structure of $T[\cdot]$.
If $T[\cdot] = [\cdot]$, then $[U]^* = U^*$ by the definition of substitution.
If $T[\cdot] = (T'[\cdot], T'')$, then by inductive hypothesis, we have $T'[U]^* = T'[U^*]^*$ and by definition, we have $(T'[U], T'')^* = T'[U]^* \otimes^{\mathsf{L}} T''^* = T'[U^*]^* \otimes^{\mathsf{L}} T''^* = (T'[U^*], T'')^*$.
The case $T[\cdot] = (T', T''[\cdot])$ is symmetric. □

Lemma 2.8. *Given a context $T[\cdot]$ and a derivation $f : A \vdash_{\mathsf{L}} B$, the following rule is admissible:*

$$\dfrac{\overset{f}{A \vdash_{\mathsf{L}} B}}{T[A]^* \vdash_{\mathsf{L}} T[B]^*}\ T[f]^*$$

Proof. The proof proceeds by induction on the structure of $T[\cdot]$.
If $T[\cdot] = [\cdot]$, then we have $T[A]^* = A$ and $T[B]^* = B$, and f is the desired derivation.
If $T[\cdot] = (T'[\cdot]; T'')$, then we construct the desired derivation as follows:

$$\dfrac{\dfrac{\dfrac{\overset{f}{A \vdash_{\mathsf{L}} B}}{T'[A]^* \vdash_{\mathsf{L}} T'[B]^*}\ T'[f]^* \quad \dfrac{}{T''^* \vdash_{\mathsf{L}} T''^*}\ \mathsf{id}}{T'[A]^* \otimes T''^* \vdash_{\mathsf{L}} T'[B]^* \otimes T''^*}\ \otimes}{(T'[A], T'')^* \vdash_{\mathsf{L}} (T'[B], T'')^*}\ \text{Lemma 2.7}$$

The case $T[\cdot] = (T', T''[\cdot])$ is symmetric. □

Definition 2.9. *We define an encoding function $[\![- \mid -]\!]$ that transforms a tree and an ordered list of formulae into a tree associating to the left:*

$$[\![T \mid [\,]]\!] = T$$
$$[\![T \mid B, \Gamma]\!] = [\![(T, B) \mid \Gamma]\!]$$

Lemma 2.10. *For any stoup S and contexts Γ and Δ, $[\![[\![S \mid \Gamma]\!] \mid \Delta]\!] = [\![S \mid \Gamma, \Delta]\!]$.*

Proof. The proof prceeds by induction on Δ.
If $\Delta = [\,]$, then $[\![[\![S \mid \Gamma]\!] \mid [\,]]\!] = [\![S \mid \Gamma]\!] = [\![S \mid \Gamma, [\,]]\!]$ by definition.
If $\Delta = (A, \Delta')$, then by Definition 2.9, inductive hypothesis, and associativity of lists, we have $[\![[\![S \mid \Gamma]\!] \mid A, \Delta']\!] = [\![[\![S \mid \Gamma, A]\!] \mid \Delta']\!] \overset{\text{I.H.}}{=} [\![S \mid (\Gamma, A), \Delta']\!] = [\![S \mid \Gamma, (A, \Delta')]\!]$. ☐

With the above lemmata, definition, and the functions $s(S)$ that maps a stoup to a tree (i.e. $s(S) = -$ if $S = -$ or $s(S) = B$ if $S = B$), we can state and prove the equivalence between `LSkG` and `LSkT`.

Theorem 2.11. *The calculi `LSkG` and `LSkT` are equivalent, meaning that the two statements below are true:*

- *For any derivation $f : S \mid \Gamma \vdash_{\mathsf{G}} C$, there exists a derivation $\mathsf{G2T}f : [\![s(S) \mid \Gamma]\!] \vdash_{\mathsf{T}} C$.*
- *For any derivation $f : T \vdash_{\mathsf{T}} C$, there exists a derivation $\mathsf{T2G}f : T^* \mid \;\vdash_{\mathsf{G}} C$.*

Proof. Both $\mathsf{G2T}$ and $\mathsf{T2G}$ are constructed by induction on height of f.

For $\mathsf{G2T}$, the interesting cases are $\otimes\mathsf{R}$ and $\multimap\mathsf{L}$. For example, if $f = \otimes\mathsf{R}(f', f'')$, then by inductive hypothesis, we have two derivations $\mathsf{G2T}\ f' : [\![s(S) \mid \Gamma]\!] \vdash_{\mathsf{T}} A$ and $\mathsf{G2T}\ f'' : [\![\mathsf{I} \mid \Delta]\!] \vdash_{\mathsf{T}} B$. Our goal sequent is $[\![[\![s(S) \mid \Gamma]\!] \mid \Delta]\!] \vdash_{\mathsf{T}} A \otimes B$, which is constructed as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathsf{G2T}\ f'}{[\![s(S) \mid \Gamma]\!] \vdash_{\mathsf{T}} A} \qquad \dfrac{\mathsf{G2T}\ f''}{[\![- \mid \Delta]\!] \vdash_{\mathsf{T}} B}}{[\![s(S) \mid \Gamma]\!], [\![- \mid \Delta]\!] \vdash_{\mathsf{T}} A \otimes B}\ \otimes\mathsf{R}}{[\![[\![s(S) \mid \Gamma]\!], - \mid \Delta]\!] \vdash_{\mathsf{T}} A \otimes B}\ \mathsf{assoc}^*}{[\![[\![s(S) \mid \Gamma]\!] \mid \Delta]\!] \vdash_{\mathsf{T}} A \otimes B}\ \mathsf{unitR}}{[\![s(S) \mid \Gamma, \Delta]\!] \vdash_{\mathsf{T}} A \otimes B}\ \text{Lemma 2.10}$$

where assoc^* means multiple applications of assoc. The case of $\multimap\mathsf{L}$ is similar.

For $\mathsf{T2G}$, the construction relies on Lemma 2.8 heavily. For example, when $f = \mathsf{unitR}\ g$, where we have $g : T[U, -] \vdash_{\mathsf{T}} C$. By inductive hypothesis, we have $\mathsf{T2G}\ g : T[U^* \otimes \mathsf{I}]^* \mid \;\vdash_{\mathsf{G}} C$. With Lemma 2.8, we construct the desired derivation as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{U^* \mid \;\vdash_{\mathsf{G}} U^*}\ \mathsf{ax} \qquad \dfrac{}{- \mid \;\vdash_{\mathsf{G}} \mathsf{I}}\ \mathsf{IR}}{U^* \mid \;\vdash_{\mathsf{G}} U^* \otimes \mathsf{I}}\ \otimes\mathsf{R}}{T[U^*]^* \mid \;\vdash_{\mathsf{G}} T[U^* \otimes \mathsf{I}]^*}\ \text{Lemma 2.8}}{T[U]^* \mid \;\vdash_{\mathsf{G}} T[U, -]^*}\ \text{Lemma 2.7} \qquad \dfrac{\mathsf{T2G}\ g}{T[U, -]^* \mid \;\vdash_{\mathsf{G}} C}}{T[U]^* \mid \;\vdash_{\mathsf{G}} C}\ \mathsf{scut}$$

The other cases are similar. ☐

3 Skew Categories

In this section, we present the definitions of left (right) skew monoidal closed categories, skew monoidal bi-closed categories, and various terms that will be used in the following section for discussion.

Definition 3.1. *A* left skew monoidal closed category $\mathbb{C}$ *is a category with a unit object* I *and two functors* $\otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ *and* $\multimap : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{C}$ *forming an adjunction* $- \otimes B \dashv B \multimap -$ *for all* B*, and three natural transformations* λ*,* ρ*,* α *typed* $\lambda_A : \mathsf{I} \otimes A \to A$*,* $\rho_A : A \to A \otimes \mathsf{I}$ *and* $\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$*, satisfying coherence conditions on morphisms due to Mac Lane [19]:*

$$\begin{array}{ccc} & \mathsf{I} \otimes \mathsf{I} & \\ {\scriptstyle \rho_{\mathsf{I}}} \nearrow & & \searrow {\scriptstyle \lambda_{\mathsf{I}}} \\ \mathsf{I} & = = = = = & \mathsf{I} \end{array} \qquad \begin{array}{ccc} (A \otimes \mathsf{I}) \otimes B & \xrightarrow{\alpha_{A,\mathsf{I},B}} & A \otimes (\mathsf{I} \otimes B) \\ {\scriptstyle \rho_A \otimes B} \uparrow & & \downarrow {\scriptstyle A \otimes \lambda_B} \\ A \otimes B & = = = = & A \otimes B \end{array}$$

$$\begin{array}{ccc} (\mathsf{I} \otimes A) \otimes B & \xrightarrow{\alpha_{\mathsf{I},A,B}} & \mathsf{I} \otimes (A \otimes B) \\ {\scriptstyle \lambda_A \otimes B} \searrow & & \swarrow {\scriptstyle \lambda_{A \otimes B}} \\ & A \otimes B & \end{array}$$

$$\begin{array}{ccc} (A \otimes B) \otimes \mathsf{I} & \xrightarrow{\alpha_{A,B,\mathsf{I}}} & A \otimes (B \otimes \mathsf{I}) \\ {\scriptstyle \rho_{A \otimes B}} \nwarrow & & \nearrow {\scriptstyle A \otimes \rho_B} \\ & A \otimes B & \end{array}$$

$$\begin{array}{ccc} (A \otimes (B \otimes C)) \otimes D & \xrightarrow{\alpha_{A,B \otimes C,D}} & A \otimes ((B \otimes C) \otimes D) \\ {\scriptstyle \alpha_{A,B,C} \otimes D} \uparrow & & \downarrow {\scriptstyle A \otimes \alpha_{B,C,D}} \\ ((A \otimes B) \otimes C) \otimes D \xrightarrow[\alpha_{A \otimes B,C,D}]{} & (A \otimes B) \otimes (C \otimes D) & \xrightarrow[\alpha_{A,B,C \otimes D}]{} A \otimes (B \otimes (C \otimes D)) \end{array}$$

Left skew monoidal closed category has other equivalent characterizations [24, 27], because natural transformations (λ, ρ, α) are in bijective correspondence with tuples of (extra)natural transformations (j, i, L) typed $j_A : \mathsf{I} \to A \multimap A$, $i_A : \mathsf{I} \multimap A \to A$, and $L_{A,B,C} : B \multimap C \to (A \multimap B) \multimap (A \multimap C)$. In particular, in a left skew *non-monoidal* closed category, (λ, ρ, α) are not available and one has to work with (j, i, L) and corresponding equations.

Definition 3.2. *A* right skew monoidal closed category $(\mathbb{C}, \mathsf{I}, \otimes, \multimap)$ *is defined with the same objects and adjoint functors as a in left skew monoidal closed category but three natural transformations* λ^{R}*,* ρ^{R}*,* α^{R} *are typed* $\lambda^{\mathsf{R}}_A : A \to \mathsf{I} \otimes A$*,* $\rho^{\mathsf{R}}_A : A \otimes \mathsf{I} \to A$ *and* $\alpha^{\mathsf{R}}_{A,B,C} : A \otimes (B \otimes C) \to (A \otimes B) \otimes C$*. The equations on morphisms are analogous but modified to fit the definition.*

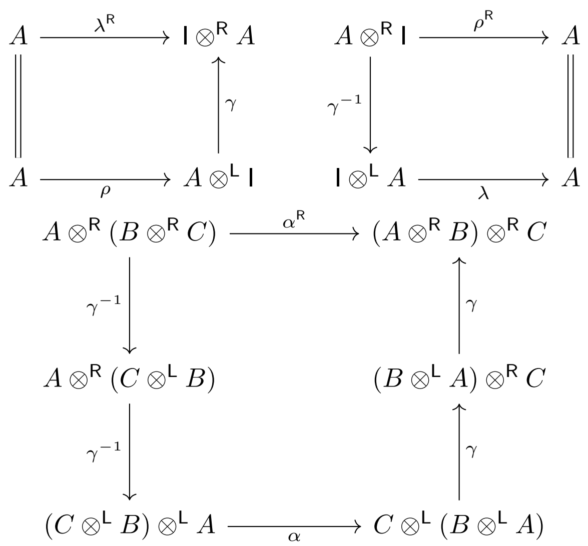
Similar to left skew monoidal closed categories, natural transformations $(\lambda^{\mathsf{R}}, \rho^{\mathsf{R}}, \alpha^{\mathsf{R}})$ are in bijective correspondence with tuples $(j^{\mathsf{R}}, i^{\mathsf{R}}, L^{\mathsf{R}})$ typed $j^{\mathsf{R}}_{A,B} : \mathbb{C}(\mathsf{I}, A \multimap B) \to \mathbb{C}(A, B)$, $i^{\mathsf{R}}_A : A \to \mathsf{I} \multimap A$, and $L^{\mathsf{R}}_{A,B,C,D} : \mathbb{C}(A, B \multimap (C \multimap D)) \to \int^X \mathbb{C}(A, X \multimap D) \times \mathbb{C}(B, C \multimap X)$, where $\int^X$ is a coend, cf. [27, Section 4], and $\mathbb{C}(A, B)$ means the set of morphisms from A to B. In parts of the next sections, where we only work with thin categories (for any two objects A and B, $\mathbb{C}(A, B)$ is either empty or a singleton set), it is safe to replace $\int^X$ with an existential quantifier.
In the rest of the paper, we usually omit subscripts of natural transformations.

Definition 3.3. *A left skew monoidal closed category is*

- associative normal *if α is a natural isomorphism;*
- left unital normal *if λ is a natural isomorphism;*
- right unital normal *if ρ is a natural isomorphism.*
- *Fully normal if α, λ, and ρ are all natural isomorphisms.*

Each normality condition can be expressed equivalently using j, i, and L. The normality conditions for right skew monoidal closed categories follow the same pattern, but with α^R, λ^R, and ρ^R instead of α, λ, and ρ.

Definition 3.4. *A category $(\mathbb{C}, \mathsf{I}, \otimes^{\mathsf{L}}, \multimap^{\mathsf{L}}, \otimes^{\mathsf{R}}, \multimap^{\mathsf{R}})$ is skew monoidal bi-closed (*SkMBiC*) if there exists a natural isomorphism $\gamma : A \otimes^{\mathsf{L}} B \to B \otimes^{\mathsf{R}} A$, $(\mathbb{C}, \mathsf{I}, \otimes^{\mathsf{L}}, \multimap^{\mathsf{L}}$) is left skew monoidal closed such that right skew structural rules are dictated by the left skew ones via γ, i.e. $\lambda^{\mathsf{R}} = \gamma \circ \rho$, $\rho^{\mathsf{R}} = \gamma^{-1} \circ \lambda$, and $\alpha^{\mathsf{R}} = \gamma \circ \gamma \circ \alpha \circ \gamma^{-1} \circ \gamma^{-1}$ diagrammatically:*



This definition combines concepts from skew bi-monoidal and bi-closed categories as introduced in [27].

In contrast to the categorical model of associative Lambek calculus, the monoidal bi-closed category, we do not have both left ($\setminus$) and right residuation ($/$), but instead have two right residuations corresponding to different tensor products. However, with the natural isomorphism γ, and selecting a specific tensor, we can simulate both left and right residuations.

In the remainder of the paper, we will develop axiomatic and sequent calculi for SkMBiC and explore its relational semantics.

4 Calculi for SkMBiC

By defining new formulae and adding rules in `LSkNL`, we can have an axiomatic calculus `SkMBiCA`, where formulae (Fma) are inductively generated by the grammar $A, B ::= X \mid \mathsf{I} \mid A \otimes^{\mathsf{L}} B \mid A \multimap^{\mathsf{L}} B \mid A \otimes^{\mathsf{R}} B \mid A \multimap^{\mathsf{R}} B$. X and I adhere to

(category laws)	$\mathsf{id} \circ f \doteq f \qquad f \doteq f \circ \mathsf{id} \qquad (f \circ g) \circ h \doteq f \circ (g \circ h)$
($\otimes^{\mathsf{L}}$ functorial)	$\mathsf{id} \otimes^{\mathsf{L}} \mathsf{id} \doteq \mathsf{id} \qquad (h \circ f) \otimes^{\mathsf{L}} (k \circ g) \doteq h \otimes^{\mathsf{L}} k \circ f \otimes^{\mathsf{L}} g$
($\multimap^{\mathsf{L}}$ functorial)	$\mathsf{id} \multimap^{\mathsf{L}} \mathsf{id} \doteq \mathsf{id} \qquad (f \circ h) \multimap^{\mathsf{L}} (k \circ g) \doteq h \multimap^{\mathsf{L}} k \circ f \multimap^{\mathsf{L}} g$
($\multimap^{\mathsf{R}}$ functorial)	$\mathsf{id} \multimap^{\mathsf{R}} \mathsf{id} \doteq \mathsf{id} \qquad (f \circ h) \multimap^{\mathsf{R}} (k \circ g) \doteq h \multimap^{\mathsf{R}} k \circ f \multimap^{\mathsf{R}} g$
(λ, ρ, α nat. trans.)	$\lambda \circ \mathsf{id} \otimes^{\mathsf{L}} f \doteq f \circ \lambda$ $\rho \circ f \doteq f \otimes^{\mathsf{L}} \mathsf{id} \circ \rho$ $\alpha \circ (f \otimes^{\mathsf{L}} g) \otimes^{\mathsf{L}} h \doteq f \otimes^{\mathsf{L}} (g \otimes^{\mathsf{L}} h) \circ \alpha$
(Mac Lane axioms)	$\lambda \circ \rho \doteq \mathsf{id} \qquad \mathsf{id} \doteq \mathsf{id} \otimes^{\mathsf{L}} \lambda \circ \alpha \circ \rho \otimes^{\mathsf{L}} \mathsf{id}$ $\lambda \circ \alpha \doteq \lambda \otimes^{\mathsf{L}} \mathsf{id} \qquad \alpha \circ \rho \doteq \mathsf{id} \otimes^{\mathsf{L}} \rho$ $\alpha \circ \alpha \doteq \mathsf{id} \otimes^{\mathsf{L}} \alpha \circ \alpha \circ \alpha \otimes^{\mathsf{L}} \mathsf{id}$
(γ isomorphism)	$\gamma \circ \gamma^{-1} \doteq \mathsf{id} \qquad \gamma^{-1} \circ \gamma \doteq \mathsf{id}$
($\pi^{(\mathsf{R})}$ nat. trans.)	$\pi f \circ g \doteq \pi(f \circ (g \otimes^{\mathsf{L}} \mathsf{id})) \qquad \pi(f \circ g) \doteq (\mathsf{id} \multimap^{\mathsf{L}} f) \circ \pi g$ $\pi(\mathsf{id} \otimes^{\mathsf{L}} f) \doteq (f \multimap^{\mathsf{L}} \mathsf{id}) \circ \pi \mathsf{id} \qquad \pi^{\mathsf{R}}(\mathsf{id} \otimes^{\mathsf{R}} f) \doteq (f \multimap^{\mathsf{R}} \mathsf{id}) \circ \pi^{\mathsf{R}} \mathsf{id}$ $\pi^{\mathsf{R}} f \circ g \doteq \pi^{\mathsf{R}}(f \circ (g \otimes^{\mathsf{R}} \mathsf{id})) \qquad \pi^{\mathsf{R}}(f \circ g) \doteq (\mathsf{id} \multimap^{\mathsf{R}} f) \circ \pi^{\mathsf{R}} g$
($\pi^{(\mathsf{R})}$ isomorphism)	$\pi(\pi^{-1} f) \doteq f \qquad \pi^{-1}(\pi f) \doteq f$ $\pi^{\mathsf{R}}(\pi^{\mathsf{R}-1} f) \doteq f \qquad \pi^{\mathsf{R}-1}(\pi^{\mathsf{R}} f) \doteq f$

Figure 1: Congruence relation on morphisms in $\mathsf{FSkMBiC}(\mathsf{At})$.

the definitions provided in Section 2, and $\otimes^{\mathsf{L}}$ and $\multimap^{\mathsf{L}}$ ($\otimes^{\mathsf{R}}$ and $\multimap^{\mathsf{R}}$) represent left (right) skew multiplicative conjunction and implication, respectively.
Derivations in `SkMBiCA` are inductively generated by the following rules:

$$\frac{}{A \vdash_{\mathsf{L}} A}\ \mathsf{id} \qquad \frac{A \vdash_{\mathsf{L}} B \quad B \vdash_{\mathsf{L}} C}{A \vdash_{\mathsf{L}} C}\ \mathsf{comp}$$

$$\frac{A \vdash_{\mathsf{L}} C \quad B \vdash_{\mathsf{L}} D}{A \otimes^{\mathsf{L}} B \vdash_{\mathsf{L}} C \otimes^{\mathsf{L}} D}\ \otimes^{\mathsf{L}}$$

$$\frac{C \vdash_{\mathsf{L}} A \quad B \vdash_{\mathsf{L}} D}{A \multimap^{\mathsf{L}} B \vdash_{\mathsf{L}} C \multimap^{\mathsf{L}} D}\ \multimap^{\mathsf{L}} \qquad \frac{C \vdash_{\mathsf{L}} A \quad B \vdash_{\mathsf{L}} D}{A \multimap^{\mathsf{R}} B \vdash_{\mathsf{L}} C \multimap^{\mathsf{R}} D}\ \multimap^{\mathsf{R}}$$

$$\frac{}{\mathsf{I} \otimes^{\mathsf{L}} A \vdash_{\mathsf{L}} A}\ \lambda \qquad \frac{}{A \vdash_{\mathsf{L}} A \otimes^{\mathsf{L}} \mathsf{I}}\ \rho \qquad \frac{}{(A \otimes^{\mathsf{L}} B) \otimes^{\mathsf{L}} C \vdash_{\mathsf{L}} A \otimes^{\mathsf{L}} (B \otimes^{\mathsf{L}} C)}\ \alpha$$

$$\frac{}{A \otimes^{\mathsf{L}} B \vdash_{\mathsf{L}} B \otimes^{\mathsf{R}} A}\ \gamma \qquad \frac{}{A \otimes^{\mathsf{R}} B \vdash_{\mathsf{L}} B \otimes^{\mathsf{L}} A}\ \gamma^{-1}$$

$$\frac{A \otimes^{\mathsf{L}} B \vdash_{\mathsf{L}} C}{A \vdash_{\mathsf{L}} B \multimap^{\mathsf{L}} C}\ \pi \qquad \frac{A \otimes^{\mathsf{R}} B \vdash_{\mathsf{L}} C}{A \vdash_{\mathsf{L}} B \multimap^{\mathsf{R}} C}\ \pi^{\mathsf{R}}$$

For any $f : A \vdash_{\mathsf{L}} B$ and $g : C \vdash_{\mathsf{L}} D$, we define $f \otimes^{\mathsf{R}} g$ as $\gamma \circ (g \otimes^{\mathsf{L}} f) \circ \gamma^{-1}$. λ^{R}, ρ^{R}, and α^{R} are also derivable.
Similar to the constructions in [30, 29, 28, 31, 26], `SkMBiCA` generates the free SkMBiC ($\mathsf{FSkMBiC}(\mathsf{At})$) over a set At in the following way:

– Objects of $\mathsf{FSkMBiC}(\mathsf{At})$ are formulae (Fma).

– Morphisms between formulae A and B are derivations of sequents $A \vdash_{\mathsf{L}} B$ and identified up to the congruence relation $\doteq$ in Figure 1: Notice that by the definition of $f \otimes^{\mathsf{R}} g$ and γ being an isomorphism, γ and γ^{-1} are natural transformations. For example, $\gamma \circ f \otimes^{\mathsf{L}} g \doteq \gamma \circ f \otimes^{\mathsf{L}} g \circ \mathsf{id} \doteq \gamma \circ f \otimes^{\mathsf{L}} g \circ \gamma^{-1} \circ \gamma = g \otimes^{\mathsf{R}} f \circ \gamma$. Similarly, naturality of $(\lambda^{\mathsf{R}}, \rho^{\mathsf{R}}, \alpha^{R})$ and the Mac Lane axioms corresponding to them hold as well.

Given a skew monoidal bi-closed category $\mathbb{D}$ with function $G : \mathsf{At} \to \mathbb{D}$, we can define functions $\overline{G}_0 : \mathsf{Fma} \to \mathbb{D}_0$ ($\mathbb{D}_0$ is the collection of objects in $\mathbb{D}$) and $\overline{G}_1 : \mathsf{FSkMBiC(At)}(A, B) \to \mathbb{D}(\overline{G}_0(A), \overline{G}_0(B))$ by induction on complexity of formulae and height of derivations respectively. This construction uniquely specifies a strict skew monoidal bi-closed functor $\overline{G} : \mathsf{FSkMBiC(At)} \to \mathbb{D}$ satisfying $\overline{G}(X) = G(X)$.

However, it remains unclear how to construct a sequent calculus à la Girard for SkMBiC. A simpler scenario to consider is the sequent calculus for right skew monoidal closed categories. In this context, recalling Definition 3.2, where natural transformations are in an opposite direction compared to left skew monoidal closed categories. One approach is to propose a dual sequent calculus to `LSkG`. Here, sequents would be of the form $\Gamma \mid S \vdash_{\mathsf{G}} A$, indicating a reversal of stoup and context, with all left rules applicable solely to the stoup. We should think of the antecedents as trees associating to the right, structured as $(A_n, (\dots, (A_1, A_0)) \dots)$. Nevertheless, $\multimap^{\mathsf{R}}$, by definition, is again a right residuation, implying that $\multimap^{\mathsf{R}}\mathsf{L}$ and $\multimap^{\mathsf{R}}\mathsf{R}$ should resemble those in `LSkG`. This requirement then necessitates contexts to appear on the right-hand side of the stoup.

Fortunately, we can develop a sequent calculus, denoted as `SkMBiCT`, which is inspired by `LSkT` to characterize SkMBiC categories. Specifically, `SkMBiCT` is an instantiation of Moortgat's multimodal Lambek calculus [20] with unit, semi-unital, and semi-associative structural rules.

Trees in `SkMBiCT` are inductively defined by the grammar $T ::= \mathsf{Fma} \mid - \mid (T, T) \mid (T; T)$. What we have defined are trees with two different ways of linking nodes: through the use of commas and semicolons, corresponding to $\otimes^{\mathsf{L}}$ and $\otimes^{\mathsf{R}}$, respectively. Contexts and substitution are defined analogously to those of `LSkT`. Sequents are in the form $T \vdash_{\mathsf{T}} A$ analogous to those in Section 2.

Derivations in `SkMBiCT` are generated recursively by the following rules:

$$\frac{}{A \vdash_{\mathsf{T}} A}\ \mathsf{ax} \qquad \frac{}{- \vdash_{\mathsf{T}} \mathsf{I}}\ \mathsf{IR} \qquad \frac{T[-] \vdash_{\mathsf{T}} C}{T[\mathsf{I}] \vdash_{\mathsf{T}} C}\ \mathsf{IL}$$

$$\frac{T[A, B] \vdash_{\mathsf{T}} C}{T[A \otimes^{\mathsf{L}} B] \vdash_{\mathsf{T}} C}\ \otimes^{\mathsf{L}}\mathsf{L} \qquad \frac{T \vdash_{\mathsf{T}} A \quad U \vdash_{\mathsf{T}} B}{T, U \vdash_{\mathsf{T}} A \otimes^{\mathsf{L}} B}\ \otimes^{\mathsf{L}}\mathsf{R}$$

$$\frac{U \vdash_{\mathsf{T}} A \quad T[B] \vdash_{\mathsf{T}} C}{T[A \multimap^{\mathsf{L}} B, U] \vdash_{\mathsf{T}} C}\ \multimap^{\mathsf{L}}\mathsf{L} \qquad \frac{T, A \vdash_{\mathsf{T}} B}{T \vdash_{\mathsf{T}} A \multimap^{\mathsf{L}} B}\ \multimap^{\mathsf{L}}\mathsf{R}$$

$$\frac{T[U_0, (U_1, U_2)] \vdash_{\mathsf{T}} C}{T[(U_0, U_1), U_2] \vdash_{\mathsf{T}} C}\ \mathsf{assoc}^{\mathsf{L}} \qquad \frac{T[U] \vdash_{\mathsf{T}} C}{T[-, U] \vdash_{\mathsf{T}} C}\ \mathsf{unitL}^{\mathsf{L}} \qquad \frac{T[U, -] \vdash_{\mathsf{T}} C}{T[U] \vdash_{\mathsf{T}} C}\ \mathsf{unitR}^{\mathsf{L}}$$

$$\frac{T[U_0, U_1] \vdash_{\mathsf{T}} C}{T[U_1; U_0] \vdash_{\mathsf{T}} C} \otimes\mathsf{comm}$$

$$\frac{T[A;B] \vdash_{\mathsf{T}} C}{T[A \otimes^{\mathsf{R}} B] \vdash_{\mathsf{T}} C} \otimes^{\mathsf{R}}\mathsf{L} \quad \frac{T \vdash_{\mathsf{T}} A \quad U \vdash_{\mathsf{T}} B}{T;U \vdash_{\mathsf{T}} A \otimes^{\mathsf{R}} B} \otimes^{\mathsf{R}}\mathsf{R}$$

$$\frac{U \vdash_{\mathsf{T}} A \quad T[B] \vdash_{\mathsf{T}} C}{T[A \multimap^{\mathsf{R}} B; U] \vdash_{\mathsf{T}} C} \multimap^{\mathsf{R}}\mathsf{L} \quad \frac{T; A \vdash_{\mathsf{T}} B}{T \vdash_{\mathsf{T}} A \multimap^{\mathsf{R}} B} \multimap^{\mathsf{R}}\mathsf{R}$$

$$\frac{T[(U_0;U_1);U_2] \vdash_{\mathsf{T}} C}{T[U_0;(U_1;U_2)] \vdash_{\mathsf{T}} C} \mathsf{assoc}^{\mathsf{R}} \quad \frac{T[U] \vdash_{\mathsf{T}} C}{T[U;-] \vdash_{\mathsf{T}} C} \mathsf{unitL}^{\mathsf{R}} \quad \frac{T[-;U] \vdash_{\mathsf{T}} C}{T[U] \vdash_{\mathsf{T}} C} \mathsf{unitR}^{\mathsf{R}}$$

We can think of these rules as originating from two separate calculi: `LSkT` (the red part with ax, IR, and IL) and another for right skew monoidal closed categories (`RSkT`, the blue part with ax, IR, and IL), linked by $\otimes\mathsf{comm}$, in other words, we can mimic all the blue rules in the style of `LSkT` (only commas appear in antecedents) and conversely, the red rules can be expressed using the blue rules. For example, we can express $\otimes^{\mathsf{R}}\mathsf{L}$, $\otimes^{\mathsf{R}}\mathsf{R}$ and $\multimap^{\mathsf{R}}\mathsf{L}$ in the style of `LSkT`:

$$\frac{T[A,B] \vdash_{\mathsf{T}} C}{T[B \otimes^{\mathsf{R}} A] \vdash_{\mathsf{T}} C} \otimes^{\mathsf{R}}\mathsf{L}' \quad = \quad \cfrac{\cfrac{T[A,B] \vdash_{\mathsf{T}} C}{T[B;A] \vdash_{\mathsf{T}} C} \otimes\mathsf{comm}}{T[B \otimes^{\mathsf{R}} A] \vdash_{\mathsf{T}} C} \otimes^{\mathsf{R}}\mathsf{L}$$

$$\frac{T \vdash_{\mathsf{T}} A \quad U \vdash_{\mathsf{T}} B}{U,T \vdash_{\mathsf{T}} A \otimes^{\mathsf{R}} B} \otimes^{\mathsf{R}}\mathsf{R}' \quad = \quad \cfrac{\cfrac{T \vdash_{\mathsf{T}} A \quad U \vdash_{\mathsf{T}} B}{T;U \vdash_{\mathsf{T}} A \otimes^{\mathsf{R}} B} \otimes^{\mathsf{R}}\mathsf{L}}{U,T \vdash_{\mathsf{T}} A \otimes^{\mathsf{R}} B} \otimes\mathsf{comm}$$

$$\frac{U \vdash_{\mathsf{T}} A \quad T[B] \vdash_{\mathsf{T}} C}{T[U, A \multimap^{\mathsf{R}} B] \vdash_{\mathsf{T}} C} \multimap^{\mathsf{R}}\mathsf{L}' \quad = \quad \cfrac{\cfrac{U \vdash_{\mathsf{T}} A \quad T[B] \vdash_{\mathsf{T}} C}{T[A \multimap^{\mathsf{R}} B; U] \vdash_{\mathsf{T}} C} \multimap^{\mathsf{R}}\mathsf{L}}{T[U, A \multimap^{\mathsf{R}} B] \vdash_{\mathsf{T}} C} \otimes\mathsf{comm}$$

$$\frac{A, T \vdash_{\mathsf{T}} B}{T \vdash_{\mathsf{T}} A \multimap^{\mathsf{R}} B} \multimap^{\mathsf{R}}\mathsf{R}' \quad = \quad \cfrac{\cfrac{A, T \vdash_{\mathsf{T}} B}{T; A \vdash_{\mathsf{T}} B} \otimes\mathsf{comm}}{T \vdash_{\mathsf{T}} A \multimap^{\mathsf{R}} B} \multimap^{\mathsf{R}}\mathsf{R}$$

Theorem 4.1. *Similar to* `LSkT`, `cut` *is admissible in* `SkMBiCT`.

$$\frac{U \vdash_{\mathsf{T}} A \quad T[A] \vdash_{\mathsf{T}} C}{T[U] \vdash_{\mathsf{T}} C} \mathsf{cut}$$

Proof. The proof proceeds similarly to that of Theorem 2.5. For the new logical rules in blue, the proofs follow the same pattern as their red counterparts. Since $\otimes\mathsf{comm}$ and all the logical and structural rules in blue are one-premise left rules, we can permute `cut` upwards. □

The equivalence between `SkMBiCA` and `SkMBiCT` can be proved by induction on height of derivations with the following admissible rules, definition, and lemmata:

Definition 4.2. *For any tree T, $T^{\#}$ is the formula obtained from T by replacing commas with $\otimes^{\mathsf{L}}$ and semicolons with $\otimes^{\mathsf{R}}$, and $-$ with I, respectively.*

Lemma 4.3. *For any context $T[\cdot]$ and tree U, $T[U]^{\#} = T[U^{\#}]^{\#}$.*

Proof. The proof proceeds by induction on the structure of $T[\cdot]$.
If $T[\cdot] = [\cdot]$, then $[U]^\# = U^\#$ by the definition of substitution.
If $T[\cdot] = (T'[\cdot], T'')$, then by inductive hypothesis, we have $T'[U]^\# = T'[U^\#]^\#$ and by the definition of $()^\#$, we have $(T'[U], T'')^\# = T'[U]^\# \otimes^{\mathsf{L}} T''^\# = T'[U^\#]^\# \otimes^{\mathsf{L}} T''^\# = (T'[U^\#], T'')^\#$.
Other cases are similar. □

Lemma 4.4. *Given a context $T[\cdot]$ and a derivation $f : A \vdash_{\mathsf{L}} B$, the following rule is admissible:*

$$\dfrac{\overset{f}{A \vdash_{\mathsf{L}} B}}{T[A]^\# \vdash_{\mathsf{L}} T[B]^\#}\ T[f]^\#$$

Proof. The proof proceeds by induction on the structure of $T[\cdot]$.
If $T[\cdot] = [\cdot]$, then we have $T[A]^\# = A$ and $T[B]^\# = B$, and f is the desired derivation.
If $T[\cdot] = (T'[\cdot]; T'')$, then we construct the desired derivation as follows:

$$\dfrac{\dfrac{\overset{f}{T'[A]^\# \vdash_{\mathsf{L}} T'[B]^\#} \qquad \dfrac{}{T''^\# \vdash_{\mathsf{L}} T''^\#}\ \mathsf{id}}{T'[A]^\# \otimes^{\mathsf{R}} T''^\# \vdash_{\mathsf{L}} T'[B]^\# \otimes^{\mathsf{R}} T''^\#}\ \otimes^{\mathsf{R}}}{(T'[A]; T'')^\# \vdash_{\mathsf{L}} (T'[B]; T'')^\#}\ \text{Lemma 4.3}$$

The case $T[\cdot] = (T'; T''[\cdot])$ is symmetric, while other cases are covered in the proof of Lemma 2.8. □

Theorem 4.5. `SkMBiCT` *is equivalent to* `SkMBiCA`*, meaning that the following two statements are true:*

1. *For any derivation $f : A \vdash_{\mathsf{L}} C$, there exists a derivation $\mathsf{A2T}f : A \vdash_{\mathsf{T}} C$.*
2. *For any derivation $f : T \vdash_{\mathsf{T}} C$, there exists a derivation $\mathsf{T2A}f : T^\# \vdash_{\mathsf{L}} C$.*

Proof. We first construct $\mathsf{A2T}$ by structural induction on the derivation f.
Case $f = \mathsf{id}$.

$$\dfrac{}{A \vdash_{\mathsf{L}} A}\ \mathsf{id} \quad \mapsto \quad \dfrac{}{A \vdash_{\mathsf{T}} A}\ \mathsf{ax}$$

Case $f = \mathsf{comp}(f', f'')$.

$$\dfrac{\overset{f'}{A \vdash_{\mathsf{L}} B} \quad \overset{f''}{B \vdash_{\mathsf{L}} C}}{A \vdash_{\mathsf{L}} C}\ \mathsf{comp} \quad \mapsto \quad \dfrac{\overset{\mathsf{A2T}f'}{A \vdash_{\mathsf{T}} B} \quad \overset{\mathsf{A2T}f''}{B \vdash_{\mathsf{T}} C}}{A \vdash_{\mathsf{T}} C}\ \mathsf{cut}$$

Case $f = \otimes^{\mathsf{L}}(f', f'')$.

$$\dfrac{\overset{f'}{A \vdash_{\mathsf{L}} C} \quad \overset{f''}{B \vdash_{\mathsf{L}} D}}{A \otimes^{\mathsf{L}} B \vdash_{\mathsf{L}} C \otimes^{\mathsf{L}} D}\ \otimes^{\mathsf{L}} \quad \mapsto \quad \dfrac{\dfrac{\overset{\mathsf{A2T}f'}{A \vdash_{\mathsf{T}} C} \quad \overset{\mathsf{A2T}f''}{B \vdash_{\mathsf{T}} D}}{A, B \vdash_{\mathsf{T}} C \otimes^{\mathsf{L}} D}\ \otimes^{\mathsf{L}}\mathsf{R}}{A \otimes^{\mathsf{L}} B \vdash_{\mathsf{T}} C \otimes^{\mathsf{L}} D}\ \otimes^{\mathsf{L}}\mathsf{L}$$

Case $f = \multimap^{\mathsf{L}} (f', f'')$.

$$\dfrac{\overset{f'}{C \vdash_{\mathsf{L}} A} \quad \overset{f''}{B \vdash_{\mathsf{L}} D}}{A \multimap^{\mathsf{L}} B \vdash_{\mathsf{L}} C \multimap^{\mathsf{L}} D}\,{\multimap^{\mathsf{L}}} \quad \mapsto \quad \dfrac{\dfrac{\overset{\mathsf{A2T}\,f'}{C \vdash_{\mathsf{T}} A} \quad \overset{\mathsf{A2T}\,f''}{B \vdash_{\mathsf{T}} D}}{A \multimap^{\mathsf{L}} B, C \vdash_{\mathsf{T}} D}\,{\multimap^{\mathsf{L}}\mathsf{L}}}{A \multimap^{\mathsf{L}} B \vdash_{\mathsf{T}} C \multimap^{\mathsf{L}} D}\,{\multimap^{\mathsf{L}}\mathsf{R}}$$

Case $f = \lambda$.

$$\dfrac{}{\mathsf{I} \otimes^{\mathsf{L}} A \vdash_{\mathsf{L}} A}\,\lambda \quad \mapsto \quad \dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash_{\mathsf{T}} A}\,\mathsf{ax}}{-, A \vdash_{\mathsf{T}} A}\,\mathsf{unitL}^{\mathsf{L}}}{\mathsf{I}, A \vdash_{\mathsf{T}} A}\,\mathsf{IL}}{\mathsf{I} \otimes^{\mathsf{L}} A \vdash_{\mathsf{T}} A}\,\otimes^{\mathsf{L}}\mathsf{L}$$

Case $f = \rho$.

$$\dfrac{}{A \vdash_{\mathsf{L}} A \otimes^{\mathsf{L}} \mathsf{I}}\,\rho \quad \mapsto \quad \dfrac{\dfrac{\dfrac{}{A \vdash_{\mathsf{T}} A}\,\mathsf{ax} \quad \dfrac{}{- \vdash_{\mathsf{T}} \mathsf{I}}\,\mathsf{IR}}{A, - \vdash_{\mathsf{T}} A \otimes^{\mathsf{L}} \mathsf{I}}\,\otimes^{\mathsf{R}}\mathsf{R}}{A \vdash_{\mathsf{T}} A \otimes^{\mathsf{L}} \mathsf{I}}\,\mathsf{unitR}^{\mathsf{L}}$$

Case $f = \alpha$.

$$\dfrac{}{(A \otimes^{\mathsf{L}} B) \otimes^{\mathsf{L}} C \vdash_{\mathsf{L}} A \otimes^{\mathsf{L}} (B \otimes^{\mathsf{L}} C)}\,\alpha$$

$$\mapsto \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash_{\mathsf{T}} A}\,\mathsf{ax} \quad \dfrac{\dfrac{}{B \vdash_{\mathsf{T}} B}\,\mathsf{ax} \quad \dfrac{}{C \vdash_{\mathsf{T}} C}\,\mathsf{ax}}{B, C \vdash_{\mathsf{T}} B \otimes^{\mathsf{L}} C}\,\otimes^{\mathsf{L}}\mathsf{R}}{A, (B, C) \vdash_{\mathsf{T}} A \otimes^{\mathsf{L}} (B \otimes^{\mathsf{L}} C)}\,\otimes^{\mathsf{L}}\mathsf{R}}{(A, B), C \vdash_{\mathsf{T}} A \otimes^{\mathsf{L}} (B \otimes^{\mathsf{L}} C)}\,\mathsf{assoc}^{\mathsf{L}}}{(A \otimes^{\mathsf{L}} B), C \vdash_{\mathsf{T}} A \otimes^{\mathsf{L}} (B \otimes^{\mathsf{L}} C)}\,\otimes^{\mathsf{L}}\mathsf{L}}{(A \otimes^{\mathsf{L}} B) \otimes^{\mathsf{L}} C \vdash_{\mathsf{T}} A \otimes^{\mathsf{L}} (B \otimes^{\mathsf{L}} C)}\,\otimes^{\mathsf{L}}\mathsf{L}$$

Case $f = \gamma$.

$$\dfrac{}{A \otimes^{\mathsf{L}} B \vdash_{\mathsf{L}} B \otimes^{\mathsf{R}} A}\,\gamma \quad \mapsto \quad \dfrac{\dfrac{\dfrac{\dfrac{}{B \vdash_{\mathsf{T}} B}\,\mathsf{ax} \quad \dfrac{}{A \vdash_{\mathsf{T}} A}\,\mathsf{ax}}{B; A \vdash_{\mathsf{T}} B \otimes^{\mathsf{R}} A}\,\otimes^{\mathsf{R}}\mathsf{R}}{A, B \vdash_{\mathsf{T}} B \otimes^{\mathsf{R}} A}\,\otimes\mathsf{comm}}{A \otimes^{\mathsf{L}} B \vdash_{\mathsf{T}} B \otimes^{\mathsf{R}} A}\,\otimes^{\mathsf{L}}\mathsf{L}$$

Case $f = \gamma^{-1}$.

$$\dfrac{}{A \otimes^{\mathsf{R}} B \vdash_{\mathsf{L}} B \otimes^{\mathsf{L}} A}\,\gamma^{-1} \quad \mapsto \quad \dfrac{\dfrac{\dfrac{\dfrac{}{B \vdash_{\mathsf{T}} B}\,\mathsf{ax} \quad \dfrac{}{A \vdash_{\mathsf{T}} A}\,\mathsf{ax}}{B, A \vdash_{\mathsf{T}} B \otimes^{\mathsf{L}} A}\,\otimes^{\mathsf{L}}\mathsf{R}}{A; B \vdash_{\mathsf{T}} B \otimes^{\mathsf{L}} A}\,\otimes\mathsf{comm}^{-1}}{A \otimes^{\mathsf{R}} B \vdash_{\mathsf{T}} B \otimes^{\mathsf{L}} A}\,\otimes^{\mathsf{R}}\mathsf{L}$$

<u>Case $f = \pi\ f'$.</u>

$$\dfrac{\overset{f'}{A \otimes^{\mathsf{L}} B \vdash_{\mathsf{L}} C}}{A \vdash_{\mathsf{L}} B \multimap^{\mathsf{L}} C}\ \pi \quad \mapsto \quad \dfrac{\dfrac{\overset{\mathsf{A2T}f'}{A \otimes^{\mathsf{L}} B \vdash_{\mathsf{T}} C}}{A, B \vdash_{\mathsf{T}} C}\ \otimes^{\mathsf{L}}\mathsf{L}^{-1}}{A \vdash_{\mathsf{T}} B \multimap^{\mathsf{L}} C}\ \multimap^{\mathsf{L}}\mathsf{R}$$

<u>Case $f = \pi^{-1}\ f'$.</u>

$$\dfrac{\overset{f'}{A \vdash_{\mathsf{L}} B \multimap^{\mathsf{L}} C}}{A \otimes^{\mathsf{L}} B \vdash_{\mathsf{L}} C}\ \pi^{-1} \quad \mapsto \quad \dfrac{\dfrac{\overset{\mathsf{A2T}f'}{A \vdash_{\mathsf{T}} B \multimap^{\mathsf{L}} C}}{A, B \vdash_{\mathsf{T}} C}\ \multimap^{\mathsf{L}}\mathsf{R}^{-1}}{A \otimes^{\mathsf{L}} B \vdash_{\mathsf{T}} C}\ \otimes^{\mathsf{L}}\mathsf{L}$$

Other cases for $\multimap^{\mathsf{R}}$ and π^{R} are similar.
We construct T2A by structural induction on f as well.
<u>Case $f = \mathsf{ax}$.</u>

$$\dfrac{}{A \vdash_{\mathsf{T}} A}\ \mathsf{ax} \quad \mapsto \quad \dfrac{}{A \vdash_{\mathsf{L}} A}\ \mathsf{id}$$

<u>Case $f = \mathsf{IR}$.</u>

$$\dfrac{}{- \vdash_{\mathsf{T}} \mathsf{I}}\ \mathsf{IR} \quad \mapsto \quad \dfrac{}{\mathsf{I} \vdash_{\mathsf{L}} \mathsf{I}}\ \mathsf{id}$$

<u>Case $f = \mathsf{IL}\ f'$.</u>

$$\dfrac{\overset{f'}{T[-] \vdash_{\mathsf{T}} C}}{T[\mathsf{I}] \vdash_{\mathsf{T}} C}\ \mathsf{IL} \quad \mapsto \quad \dfrac{\overset{\mathsf{T2A}f'}{T[-]^{\#} \vdash_{\mathsf{L}} C}}{\overline{T[\mathsf{I}]^{\#} \vdash_{\mathsf{L}} C}}$$

<u>Case $f = \otimes\mathsf{comm}\ f'$</u>

$$\dfrac{\overset{f'}{T[U_0, U_1] \vdash_{\mathsf{T}} C}}{T[U_1; U_0] \vdash_{\mathsf{T}} C}\ \otimes\mathsf{comm}$$

$$\mapsto \quad \dfrac{\dfrac{\dfrac{\dfrac{}{U_1^{\#} \otimes^{\mathsf{R}} U_0^{\#} \vdash_{\mathsf{L}} U_0^{\#} \otimes^{\mathsf{L}} U_1^{\#}}\ \gamma^{-1}}{T[U_1^{\#} \otimes^{\mathsf{R}} U_0^{\#}]^{\#} \vdash_{\mathsf{L}} T[U_0^{\#} \otimes^{\mathsf{L}} U_1^{\#}]^{\#}}\ \text{Lemma 4.4}}{T[U_1; U_0]^{\#} \vdash_{\mathsf{L}} T[U_0, U_1]^{\#}}\ \text{Lemma 4.3} \quad \overset{\mathsf{T2Af'}}{T[U_0, U_1]^{\#} \vdash_{\mathsf{L}} C}}{T[U_1; U_0]^{\#} \vdash_{\mathsf{L}} C}\ \mathsf{comm}$$

<u>Case $f = \otimes\mathsf{L}\ f'$</u>

$$\dfrac{\overset{f'}{T[A, B] \vdash_{\mathsf{T}} C}}{T[A \otimes^{\mathsf{L}} B] \vdash_{\mathsf{T}} C}\ \otimes^{\mathsf{L}}\mathsf{L} \quad \mapsto \quad \dfrac{\overset{\mathsf{T2A}f'}{T[A, B]^{\#} \vdash_{\mathsf{L}} C}}{\overline{T[A \otimes^{\mathsf{L}} B]^{\#} \vdash_{\mathsf{L}} C}}$$

Case $f = \otimes^{\mathsf{L}}\mathsf{R}(f', f'')$.

$$\frac{\overset{f'}{T \vdash_{\mathsf{T}} A} \quad \overset{f''}{U \vdash_{\mathsf{T}} B}}{T, U \vdash_{\mathsf{T}} A \otimes^{\mathsf{L}} B}\ \otimes^{\mathsf{L}}\mathsf{R} \quad\mapsto\quad \frac{\dfrac{\overset{\mathsf{T2A}f'}{T^{\#} \vdash_{\mathsf{L}} A} \quad \overset{\mathsf{T2A}f''}{U^{\#} \vdash_{\mathsf{L}} B}}{T^{\#} \otimes^{\mathsf{L}} U^{\#} \vdash_{\mathsf{L}} A \otimes^{\mathsf{L}} B}\ \otimes^{\mathsf{L}}}{(T, U)^{\#} \vdash_{\mathsf{L}} A \otimes^{\mathsf{L}} B}\ \text{Lemma 4.3}$$

Case $f = \multimap^{\mathsf{L}}\mathsf{L}$.

$$\frac{\overset{f'}{U \vdash_{\mathsf{T}} A} \quad \overset{f''}{T[B] \vdash_{\mathsf{T}} C}}{T[A \multimap^{\mathsf{L}} B, U] \vdash_{\mathsf{T}} C}\ \multimap^{\mathsf{L}}\mathsf{L}$$

$$\mapsto\ \frac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \multimap^{\mathsf{L}} B \vdash_{\mathsf{L}} A \multimap^{\mathsf{L}} B}\ \mathsf{id} \quad \overset{\mathsf{T2A}f'}{U^{\#} \vdash_{\mathsf{L}} A}}{(A \multimap^{\mathsf{L}} B) \otimes^{\mathsf{L}} U^{\#} \vdash_{\mathsf{L}} (A \multimap^{\mathsf{L}} B) \otimes^{\mathsf{L}} A}\ \otimes^{\mathsf{L}}}{T[(A \multimap^{\mathsf{L}} B) \otimes^{\mathsf{L}} U^{\#}]^{\#} \vdash_{\mathsf{L}} T[(A \multimap^{\mathsf{L}} B) \otimes^{\mathsf{L}} A]^{\#}}\ \text{Lemma 4.4} \quad \dfrac{\dfrac{\dfrac{}{A \multimap^{\mathsf{L}} B \vdash_{\mathsf{L}} A \multimap^{\mathsf{L}} B}\ \mathsf{id}}{(A \multimap^{\mathsf{L}} B) \otimes^{\mathsf{L}} A \vdash_{\mathsf{L}} B}\ \pi^{-1}}{T[(A \multimap^{\mathsf{L}} B) \otimes^{\mathsf{L}} A]^{\#} \vdash_{\mathsf{L}} T[B]^{\#}}\ \text{Lemma 4.4}}{\dfrac{T[(A \multimap^{\mathsf{L}} B) \otimes^{\mathsf{L}} U^{\#}]^{\#} \vdash_{\mathsf{L}} T[B]^{\#}}{T[(A \multimap^{\mathsf{L}} B), U]^{\#} \vdash_{\mathsf{L}} T[B]^{\#}}\ \text{Lemma 4.3}}\ \mathsf{comp} \quad \overset{\mathsf{T2A}f''}{T[B]^{\#} \vdash_{\mathsf{L}} C}}{T[(A \multimap^{\mathsf{L}} B), U^{\#}]^{\#} \vdash_{\mathsf{L}} C}\ \mathsf{comp}$$

Case $f = \multimap^{\mathsf{L}}\mathsf{R}\ f'$

$$\frac{\overset{f'}{T, A \vdash_{\mathsf{T}} B}}{T \vdash_{\mathsf{T}} A \multimap^{\mathsf{L}} B}\ \multimap^{\mathsf{L}}\mathsf{R} \quad\mapsto\quad \frac{\overset{\mathsf{T2A}f'}{T^{\#} \otimes^{\mathsf{L}} A \vdash_{\mathsf{L}} B}}{T^{\#} \vdash_{\mathsf{L}} A \multimap^{\mathsf{L}} B}\ \pi$$

Case $f = \mathsf{assoc}^{\mathsf{L}}\ f'$

$$\frac{\overset{f'}{T[U_0, (U_1, U_2)] \vdash_{\mathsf{T}} C}}{T[(U_0, U_1), U_2] \vdash_{\mathsf{T}} C}\ \mathsf{assoc}^{\mathsf{L}}$$

$$\mapsto\ \frac{\dfrac{\dfrac{\dfrac{}{(U_0^{\#} \otimes^{\mathsf{L}} U_1^{\#}) \otimes^{\mathsf{L}} U_2^{\#} \vdash_{\mathsf{L}} U_0^{\#} \otimes^{\mathsf{L}} (U_1^{\#} \otimes^{\mathsf{L}} U_2^{\#})}\ \alpha}{T[(U_0^{\#} \otimes^{\mathsf{L}} U_1^{\#}) \otimes^{\mathsf{L}} U_2^{\#}]^{\#} \vdash_{\mathsf{L}} T[U_0^{\#} \otimes^{\mathsf{L}} (U_1^{\#} \otimes^{\mathsf{L}} U_2^{\#})]^{\#}}\ \text{Lemma 4.4}}{T[(U_0, U_1), U_2]^{\#} \vdash_{\mathsf{L}} T[U_0, (U_1, U_2)]^{\#}}\ \text{Lemma 4.3} \quad \overset{\mathsf{T2A}f'}{T[U_0, (U_1, U_2)]^{\#} \vdash_{\mathsf{L}} C}}{T[(U_0, U_1), U_2]^{\#} \vdash_{\mathsf{T}} C}\ \mathsf{comp}$$

Case $f = \mathsf{unitL}^{\mathsf{L}}\ f'$

$$\frac{\overset{f'}{T[U] \vdash_{\mathsf{T}} C}}{T[-, U] \vdash_{\mathsf{T}} C}\ \mathsf{unitL}^{\mathsf{L}}$$

$$\mapsto\ \frac{\dfrac{\dfrac{\dfrac{}{\mathsf{I} \otimes^{\mathsf{L}} U^{\#} \vdash_{\mathsf{L}} U^{\#}}\ \lambda}{T[\mathsf{I} \otimes^{\mathsf{L}} U^{\#}]^{\#} \vdash_{\mathsf{L}} T[U^{\#}]^{\#}}\ \text{Lemma 4.4}}{T[-, U]^{\#} \vdash_{\mathsf{L}} T[U]^{\#}}\ \text{Lemma 4.3} \quad \overset{\mathsf{T2A}f'}{T[U]^{\#} \vdash_{\mathsf{T}} C}}{T[-, U]^{\#} \vdash_{\mathsf{L}} C}\ \mathsf{comp}$$

Case $f = \mathsf{unitR^L}\ f'$

$$\dfrac{\overset{f'}{T[U,-] \vdash_{\mathsf{T}} C}}{T[U] \vdash_{\mathsf{T}} C}\ \mathsf{unitR^L}$$

$$\mapsto\quad \dfrac{\dfrac{\dfrac{\dfrac{}{U^{\#} \vdash_{\mathsf{L}} U^{\#} \otimes^{\mathsf{L}} \mathsf{I}}\ \rho}{T[U^{\#}]^{\#} \vdash_{\mathsf{L}} T[U^{\#} \otimes^{\mathsf{L}} \mathsf{I}]^{\#}}\ \text{Lemma 4.4}}{T[U]^{\#} \vdash_{\mathsf{L}} T[U,-]^{\#}}\ \text{Lemma 4.3} \qquad \overset{\mathsf{T2A}f'}{T[U,-]^{\#} \vdash_{\mathsf{T}} C}}{T[U]^{\#} \vdash_{\mathsf{L}} C}\ \mathsf{comp}$$

Other cases for right skew rules are similar. □

5 Relational Semantics of SkMBiCA and Application

In this section, we present the relational semantics of SkMBiCA. Furthermore, the relational semantics for SkMBiCA is characterized modularly, allowing us to construct models for semi-substructural logics step by step by incorporating additional structural conditions into the frame. The modularity allows us to provide an algebraic proof for the main theorems concerning the interdefinability of a series of skew categories as discussed in [27].

A preordered ternary frame with a special subset is $\langle W, \leq, \mathbb{I}, \mathbb{L}\rangle$, where W is a set, $\leq$ is a preorder relation on W, $\mathbb{I}$ is a downwards closed subset of W, and $\mathbb{L}$ is an arbitrary ternary relation on W, where $\mathbb{L}$ is upwards closed in the first two arguments and downwards closed in the last argument with respect to $\leq$. For example, given $\mathbb{L}abc$, if we have $a \leq a'$, $b \leq b'$, and $c' \leq c$, then $\mathbb{L}a'b'c'$.

Definition 5.1. *We list properties of ternary relations which we will focus on.*

Left Skew Associativity (LSA)	$\forall a, b, c, d, x \in W, \mathbb{L}abx\ \&\ \mathbb{L}xcd \longrightarrow \exists y \in W$ *such that* $\mathbb{L}bcy\ \&\ \mathbb{L}ayd$.
Left Skew Left Unitality (LSLU)	$\forall a, b \in W, e \in \mathbb{I}, \mathbb{L}eab \longrightarrow b \leq a$.
Left Skew Right Unitality (LSRU)	$\forall a \in W, \exists e \in \mathbb{I}$ *such that* $\mathbb{L}aea$.
Right Skew Associativity (RSA)	$\forall a, b, c, d, x \in W, \mathbb{L}bcx\ \&\ \mathbb{L}axd \longrightarrow \exists y \in W$ *such that* $\mathbb{L}aby\ \&\ \mathbb{L}ycd$.
Right Skew Left Unitality (RSLU)	$\forall a \in W, \exists e \in \mathbb{I}$ *such that* $\mathbb{L}eaa$.
Right Skew Right Unitality (RSRU)	$\forall a, b \in W, e \in \mathbb{I}, \mathbb{L}aeb \longrightarrow b \leq a$.

Given another ternary relation $\mathbb{R}$, we define

$$\mathbb{L}\mathbb{R}\text{-reverse} \quad \forall a, b, c \in W, \mathbb{L}abc \longleftrightarrow \mathbb{R}bac.$$

The associativity and unitality conditions are adapted from the theory of relational monoids [23] and relational semantics for Lambek calculus [12].

A $\texttt{SkMBiCA}$ frame is a quintuple $\langle W, \leq, \mathbb{I}, \mathbb{L}, \mathbb{R}\rangle$, where $\mathbb{LR}$-reverse is satisfied, $\mathbb{L}$ satisfies LSA, LSLU, LSRU, and $\mathbb{R}$ automatically satisfies RSA, RSLU, RSRU because of $\mathbb{LR}$-reverse.

Unlike studies in $\texttt{NL}$ e.g. [12, 20, 22], where two associativity conditions simultaneously hold for a relation or not, we explore two relations where one satisfies LSA and the other satisfies RSA. Another distinction from the existing studies on semantics for $\texttt{NL}$ with unit [9] (or non-commutative linear logic [1]) is that while W is commonly assumed to be an unital groupoid (or monoid in the case of linear logic), here, we should consider that the unit behaves differently for different relations.

We denote the set of downwards closed subsets of W as $\mathcal{P}_{\downarrow}(W)$.

Definition 5.2. *A function $v : \mathsf{Fma} \to \mathcal{P}_{\downarrow}(W)$ on a $\texttt{SkMBiCA}$ frame is a valuation if it satisfies:*

$$\begin{array}{ll} v(\mathsf{I}) & = \mathbb{I} \\ v(A \otimes^{\mathsf{L}} B) & = \{c : \exists a \in v(A), b \in v(B),\ \mathbb{L}abc\} \\ v(A \multimap^{\mathsf{L}} B) & = \{c : \forall a \in v(A), b \in W,\ \mathbb{L}cab \Rightarrow b \in v(B)\} \\ v(A \otimes^{\mathsf{R}} B) & = \{c : \exists a \in v(A), b \in v(B),\ \mathbb{R}abc\} \\ v(A \multimap^{\mathsf{R}} B) & = \{c : \forall a \in v(A), b \in W,\ \mathbb{R}cab \Rightarrow b \in v(B)\} \end{array}$$

We define a $\texttt{SkMBiCA}$ model to be a $\texttt{SkMBiCA}$ frame with a valuation function, i.e. $\langle W, \leq, \mathbb{I}, \mathbb{L}, \mathbb{R}, v\rangle$. A sequent $A \vdash_{\mathsf{L}} B$ is valid in a model $\langle W, \leq, \mathbb{I}, \mathbb{L}, \mathbb{R}, v\rangle$ if $v(A) \subseteq v(B)$ and is valid in a frame if for any v for that frame, $v(A) \subseteq v(B)$.

Theorem 5.3 (Soundness)**.** *If a sequent $A \vdash_{\mathsf{L}} B$ is provable in $\texttt{SkMBiCA}$ then it is valid in any $\texttt{SkMBiCA}$ model.*

Proof. The proof is adapted from [12, 22], where the cases of α and α^{R} have been discussed. Therefore, we only elaborate on new cases arising in $\texttt{SkMBiCA}$.

- If the derivation is the axiom $\lambda : \mathsf{I} \otimes^{\mathsf{L}} A \vdash_{\mathsf{L}} A$, then for any $\texttt{SkMBiCA}$ model $\langle W, \mathbb{I}, \mathbb{L}, \mathbb{R}, v\rangle$ and any $a \in v(\mathsf{I} \otimes^{\mathsf{L}} A)$, there exist $e \in \mathbb{I}$, $a' \in v(A)$, and $\mathbb{L}ea'a$. By LSLU, we know that $a \leq a'$, and then $a \in v(A)$.
- If the derivation is the axiom $\rho : A \vdash_{\mathsf{L}} A \otimes^{\mathsf{L}} \mathsf{I}$, then for any $\texttt{SkMBiCA}$ model $\langle W, \mathbb{I}, \mathbb{L}, \mathbb{R}, v\rangle$ and any $a \in v(A)$, by LSRU, there exists $e \in \mathbb{I}$ such that $\mathbb{L}aea$, which means that $a \in v(A \otimes^{\mathsf{L}} \mathsf{I})$.
- If the derivation is the axiom $\gamma : A \otimes^{\mathsf{L}} B \vdash_{\mathsf{L}} B \otimes^{\mathsf{R}} A$, then for any $\texttt{SkMBiCA}$ model $\langle W, \mathbb{I}, \mathbb{L}, \mathbb{R}, v\rangle$ and any $c \in v(A \otimes^{\mathsf{L}} B)$, there exist $a \in v(A)$ and $b \in v(B)$ such that $\mathbb{L}abc$. By $\mathbb{LR}$-reverse, we have $\mathbb{R}bac$, therefore $c \in v(B \otimes^{\mathsf{R}} A)$.
- The case of γ^{-1} is similar.

□

Definition 5.4. *The canonical model of $\texttt{SkMBiCA}_{\mathsf{e}}$ is $\langle W, \leq, \mathbb{I}, \mathbb{L}, \mathbb{R}, v\rangle$ where*

- *$W = \mathsf{Fma}$ and $A \leq B$ if and only if $A \vdash_{\mathsf{L}} B$,*
- *$\mathbb{I} = v(\mathsf{I})$,*
- *$\mathbb{L}ABC$ if and only if $C \vdash_{\mathsf{L}} A \otimes^{\mathsf{L}} B$,*

– $\mathbb{R}ABC$ *if and only if* $C \vdash_{\mathsf{L}} A \otimes^{\mathsf{R}} B$, *and*

– $v(A) = \{B \mid B \vdash_{\mathsf{L}} A$ *is provable in* `SkMBiCA`$\}$.

Lemma 5.5. *The canonical model is a* `SkMBiCA` *model.*

Proof.

– The set $(\mathsf{Fma}, \vdash_{\mathsf{L}})$ is a preorder because of the rules id and comp, and the set $\mathbb{I}$ is downwards closed because of comp. The relations $\mathbb{L}$ and $\mathbb{R}$ are downwards closed in their last argument because of the rule comp. They are upwards closed in their first two arguments due to the rules $\otimes^{\mathsf{L}}$ and $\otimes^{\mathsf{R}}$, respectively. These facts ensure that $\langle \mathsf{Fma}, \vdash_{\mathsf{L}}, \mathbb{I}, \mathbb{L}, \mathbb{R} \rangle$ is a ternary frame.

– We show two cases (LSRU and LSRU) of the proof that $\mathbb{L}, \mathbb{R}$ satisfy their corresponding conditions, while other cases are similar.

(LSLU) Given any two formulae A and B, and $J \in \mathbb{I}$ with $\mathbb{L}JAB$, we have $J \vdash_{\mathsf{L}} \mathsf{I}$, and $B \vdash_{\mathsf{L}} J \otimes^{\mathsf{L}} A$, then we can construct $B \vdash_{\mathsf{L}} A$ as follows:

$$\dfrac{\dfrac{B \vdash_{\mathsf{L}} J \otimes^{\mathsf{L}} A \quad \dfrac{J \vdash_{\mathsf{L}} \mathsf{I} \quad \dfrac{}{A \vdash_{\mathsf{L}} A}\,\mathsf{id}}{J \otimes^{\mathsf{L}} A \vdash_{\mathsf{L}} \mathsf{I} \otimes^{\mathsf{L}} A}\,\otimes^{\mathsf{L}}}{B \vdash_{\mathsf{L}} \mathsf{I} \otimes^{\mathsf{L}} A}\,\mathsf{comp} \quad \dfrac{}{\mathsf{I} \otimes^{\mathsf{L}} A \vdash_{\mathsf{L}} A}\,\lambda}{B \vdash_{\mathsf{L}} A}\,\mathsf{comp}$$

(LSRU) By the axiom ρ, for any formula A, we have $A \vdash_{\mathsf{L}} A \otimes^{\mathsf{L}} \mathsf{I}$, i.e. $\mathbb{L}AIA$.

– The valuation v is downwards closed because of the rule comp. The other conditions on connectives are satisfied by definition.

Therefore, $\langle \mathsf{Fma}, \vdash_{\mathsf{L}}, \mathbb{I}, \mathbb{L}, \mathbb{R}, v \rangle$ is a `SkMBiCA` model. ☐

Theorem 5.6 (Completeness)**.** *If* $A \vdash_{\mathsf{L}} B$ *is valid in any* `SkMBiCA` *model, then it is provable in* `SkMBiCA`.

Proof. If $A \vdash_{\mathsf{L}} B$ is valid in any `SkMBiCA` model, then it is valid in the canonical model, i.e. $v(A) \subseteq v(B)$ in the canonical model. From $A \vdash_{\mathsf{L}} A$, by definition of v, we have $A \in v(A)$, and because $v(A) \subseteq v(B)$, we know that $A \in v(B)$, therefore $A \vdash_{\mathsf{L}} B$. ☐

We show a correspondence between frame conditions and the validity of structural laws in frames.

Theorem 5.7. *For any ternary frame* $\langle W, \leq, \mathbb{I}, \mathbb{L}, \mathbb{R} \rangle$,

	$\mathbb{LR}$*-reverse holds*	$\longleftrightarrow$	γ *and* γ^{-1} *valid*	
$\alpha^{(\mathsf{R})}$ *valid*	$\longleftrightarrow$	*LSA (RSA) holds*	$\longleftrightarrow$	$L^{(\mathsf{R})}$ *valid*
$\lambda^{(\mathsf{R})}$ *valid*	$\longleftrightarrow$	*LSLU (RSLU) holds*	$\longleftrightarrow$	$j^{(\mathsf{R})}$ *valid*
$\rho^{(\mathsf{R})}$ *valid*	$\longleftrightarrow$	*LSRU (RSRU) holds*	$\longleftrightarrow$	$i^{(\mathsf{R})}$ *valid*

Proof. The first case is that $\mathbb{LR}$-reverse holds if and only if γ and γ^{-1} are valid, i.e. $v(A \otimes^{\mathsf{L}} B) = v(B \otimes^{\mathsf{R}} A)$.

($\longrightarrow$) For any $x \in v(A \otimes^{\mathsf{L}} B) \subseteq W$, there exists $a \in v(A), b \in v(B)$ and $\mathbb{L}abx$. By $\mathbb{LR}$-reverse, we have $\mathbb{R}bax$ meaning that $x \in v(B \otimes^{\mathsf{R}} A)$. The other way around is similar.

($\longleftarrow$) Suppose that for any v, A, B, we have $v(A \otimes^{\mathsf{L}} B) = v(B \otimes^{\mathsf{R}} A)$. Consider any $a, b, x \in W$ such that $\mathbb{L}abx$. We take $v(A) = a{\downarrow}$ and $v(B) = b{\downarrow}$ for some $A, B \in \mathsf{At}$. By the definition of v and assumption, x belongs to $v(A \otimes^{\mathsf{L}} B)$ which is equal to $v(B \otimes^{\mathsf{R}} A)$, therefore $\mathbb{R}bax$. The other direction is similar.

λ : LSLU holds if and only if λ is valid.

($\longrightarrow$) This is similar to case of λ in the proof of Theorem 5.3.

($\longleftarrow$) Suppose that λ is valid, i.e. for any A and v, we have $v(\mathsf{I} \otimes^{\mathsf{L}} A) \subseteq v(A)$. Consider any $a, b \in W$, $e \in \mathbb{I}$ such that $\mathbb{L}eab$. We take $v(A) = a{\downarrow}$ for some $A \in \mathsf{At}$. By $\mathbb{L}eab$ and the assumption, we know that $b \in v(A)$, which means that $b \leq a$.

ρ : LSRU holds if and only if ρ is valid.

($\longrightarrow$) This is similar to case of ρ in the proof of Theorem 5.3.

($\longleftarrow$) Suppose ρ is valid, i.e. for any A and v, $v(A) \subseteq v(A \otimes^{\mathsf{L}} \mathsf{I})$. Consider any $a \in W$. We take $v(A) = a{\downarrow}$ for some $A \in \mathsf{At}$. By the assumption, there exist $a' \in v(A)$ and $e \in \mathbb{I}$ such that $\mathbb{L}a'ea$. Because $\mathbb{L}$ is upwards closed in its first argument, we know that $\mathbb{L}aea$.

α : LSA holds if and only if α is valid.

($\longrightarrow$) For any $s \in v((A \otimes^{\mathsf{L}} B) \otimes^{\mathsf{L}} C)$, there exists $a \in v(A), b \in v(B), x \in v(A \otimes^{\mathsf{L}} B), c \in v(C), \mathbb{L}abx$, and $\mathbb{L}xcs$. By LSA, there exists $y \in W$ such that $\mathbb{L}bcy$ and $\mathbb{L}ays$, then by definition of v, $y \in v(B \otimes^{\mathsf{L}} C)$ and $s \in v(A \otimes^{\mathsf{L}} (B \otimes^{\mathsf{L}} C))$.

($\longleftarrow$) Suppose that α is valid, i.e. for any A, B, C, v, we have $v((A \otimes^{\mathsf{L}} B) \otimes^{\mathsf{L}} C) \subseteq v(A \otimes^{\mathsf{L}} (B \otimes^{\mathsf{L}} C))$. Consider any $a, b, x, c, d \in W$ such that $\mathbb{L}abx$ and $\mathbb{L}xcd$. We take $v(A) = a{\downarrow}, v(B) = b{\downarrow}$, $v(C) = c{\downarrow}$ for some $A, B, C \in \mathsf{At}$, then we know that $x \in v(A \otimes^{\mathsf{L}} B)$ and $d \in v((A \otimes^{\mathsf{L}} B) \otimes^{\mathsf{L}} C)$. By the assumption, d belongs to $v(A \otimes^{\mathsf{L}} (B \otimes^{\mathsf{L}} C))$ as well, which means that there exist $a', b', y, c' \in W$ such that $\mathbb{L}b'c'y$ and $\mathbb{L}a'yd$. Because $\mathbb{L}$ is upwards closed in its first and second arguments, we have $\mathbb{L}bcy$ and $\mathbb{L}ayd$ as desired.

L : LSA holds if and only if for any A, B, C and v, $v(B \multimap^{\mathsf{L}} C) \subseteq v((A \multimap^{\mathsf{L}} B) \multimap^{\mathsf{L}} (A \multimap^{\mathsf{L}} C))$.

($\longrightarrow$) For any $s \in v(B \multimap^{\mathsf{L}} C)$, we show $s \in v((A \multimap^{\mathsf{L}} B) \multimap^{\mathsf{L}} (A \multimap^{\mathsf{L}} C))$. By definition, from assumptions $x \in v(A \multimap^{\mathsf{L}} B)$, $\mathbb{L}sxy$, $y \in v(A \multimap^{\mathsf{L}} C)$, $a \in A$, $c \in W$, and $\mathbb{L}yac$, we have to prove that $c \in C$. By LSA, there exists $x' \in W$ such that $\mathbb{L}xax'$ and $\mathbb{L}sx'c$. We get $x' \in B$ due to $x \in v(A \multimap^{\mathsf{L}} B)$. Thus, we have $c \in C$ because $s \in v(B \multimap^{\mathsf{L}} C)$.

($\longleftarrow$) Suppose that for any A, B, C and v, we have $v(B \multimap^{\mathsf{L}} C) \subseteq v((A \multimap^{\mathsf{L}} B) \multimap^{\mathsf{L}} (A \multimap^{\mathsf{L}} C))$. Consider $a, b, x, c, d \in W$ such that $\mathbb{L}abx$ and

$\mathbb{L}xcd$. Take $v(A) = c\downarrow$, $v(B) = \{y \mid \mathbb{L}bcy\}$, and $v(C) = \{d' \mid \exists y \in v(B), \mathbb{L}ayd'\}$ for some $A, B, C \in \mathsf{At}$. Given any $y \in v(B)$ and any $d' \in W$, if $\mathbb{L}ayd'$, then by definition of $v(C)$, $d' \in v(C)$, therefore $a \in v(B \multimap^{\mathsf{L}} C)$. By assumption, $a \in v((A \multimap^{\mathsf{L}} B) \multimap^{\mathsf{L}} (A \multimap^{\mathsf{L}} C))$ as well, which means that, for any $b' \in v(A \multimap^{\mathsf{L}} B)$, $x' \in W$, $c' \in v(A)$ and $d' \in W$, if $\mathbb{L}ab'x'$, then $x' \in v(A \multimap^{\mathsf{L}} C)$, and if $\mathbb{L}x'c'd'$, then $d' \in C$. By the definition of $v(B)$ and assumptions $\mathbb{L}abx$ and $\mathbb{L}xcd$, we have $b \in v(A \multimap^{\mathsf{L}} B)$, $x \in v(A \multimap^{\mathsf{L}} C)$, therefore $d \in v(C)$, which means that there exists $y \in W$ such that $\mathbb{L}bcy$ and $\mathbb{L}ayd$.

j^{R} : RSLU holds if and only if for any A, B and v, if $\mathbb{I} \subseteq v(A \multimap^{\mathsf{R}} B)$, then $v(A) \subseteq v(B)$.

($\longrightarrow$) By RSLU, for all $a \in v(A)$, there exists $e \in \mathbb{I}$ such that $\mathbb{R}eaa$, then we have $a \in v(B)$ because $e \in v(A \multimap^{\mathsf{R}} B)$.

($\longleftarrow$) Suppose that for any A, B and v, if $\mathbb{I} \subseteq v(A \multimap^{\mathsf{R}} B)$, then $v(A) \subseteq v(B)$. Consider any $a \in W$. We take $v(A) = a\downarrow$ and $v(B) = \{b \mid \exists e \in \mathbb{I}, \mathbb{R}eab\}$ for some $A, B \in \mathsf{At}$. For any $e' \in \mathbb{I}$, $a' \in v(A)$, and $b' \in W$, if $\mathbb{R}e'a'b'$, then because $\mathbb{R}$ is upwards closed in its second argument, we have $b' \in v(B)$, which means $e' \in v(A \multimap^{\mathsf{R}} B)$. Therefore $\mathbb{I} \subseteq v(A \multimap^{\mathsf{R}} B)$. From the assumption, we can now conclude that $v(A) \subseteq v(B)$. In particular, $a \in v(B)$, which means that there exists $e \in \mathbb{I}$ such that $\mathbb{R}eaa$.

L^{R} : RSA holds if and only if for any A, B, C, D and v, if $v(A) \subseteq v(B \multimap^{\mathsf{R}} (C \multimap^{\mathsf{R}} D))$ then there exists X such that $v(A) \subseteq v(X \multimap^{\mathsf{R}} D)$ and $v(B) \subseteq v(C \multimap^{\mathsf{R}} X)$.

($\longrightarrow$) We expand the assumption.
For any A, B, C, D, $a \in v(A)$, and $b, z \in W$, if $b \in v(B)$ and $\mathbb{R}abz$ then $z \in v(C \multimap^{\mathsf{R}} D)$ and for all $z \in v(C \multimap^{\mathsf{R}} D)$, for all $c, d \in W$ if $c \in v(C)$ and $\mathbb{R}zcd$, then $d \in v(D)$. In other words, for any $z, d \in W$, if there are $a \in v(A)$, $b \in v(B)$, $c \in v(C)$, $\mathbb{R}abz$, and $\mathbb{R}zcd$, then $d \in v(D)$.
We take $X = B \otimes^{\mathsf{R}} C$ and show it satisfies the two following statements:

- For any $a \in v(A)$, we show that $a \in v((B \otimes^{\mathsf{R}} C) \multimap^{\mathsf{R}} D)$. For any $x \in v(B \otimes^{\mathsf{R}} C)$ and $d \in W$, if $\mathbb{R}axd$, then by definition of $\otimes^{\mathsf{R}}$, we have $\mathbb{R}bcx$, where $b \in v(B)$ and $c \in v(C)$. By RSA, there exists $z \in W$ such that $\mathbb{R}abz$, and $\mathbb{R}zcd$. By the expanded assumption, $d \in v(D)$. Therefore $a \in v((B \otimes^{\mathsf{R}} C) \multimap^{\mathsf{R}} D)$.
- For any $b \in v(B)$, $c \in v(C)$, and $x \in W$, suppose $\mathbb{R}bcx$, then $x \in v(B \otimes^{\mathsf{R}} C)$ by definition of $\otimes^{\mathsf{R}}$. Therefore $b \in v(C \multimap^{\mathsf{R}} (B \otimes^{\mathsf{R}} C))$.

($\longleftarrow$) Assume that for any A, B, C, D and v, if $v(A) \subseteq v(B \multimap^{\mathsf{R}} (C \multimap^{\mathsf{R}} D))$, then there exists X such that $v(A) \subseteq v(X \multimap^{\mathsf{R}} D)$ and $v(B) \subseteq v(C \multimap^{\mathsf{R}} X)$. Suppose that we have $a, b, c, d, x \in W$ such that $\mathbb{R}axd$ and $\mathbb{R}bcx$, then we take $v(A) = a\downarrow$, $v(B) = b\downarrow$, $v(C) = c\downarrow$, and $v(D) = \{d' \mid \exists y, \mathbb{R}aby \& \mathbb{R}ycd'\}$ for some $A, B, C, D \in \mathsf{At}$. For any $a' \in v(A)$, given any $b' \in v(B)$, $x' \in W$, $c' \in v(C)$, $d' \in W$ such that $\mathbb{R}a'b'x'$ and $\mathbb{R}x'c'd'$. Because $\mathbb{R}$ is upwards closed in its first and second arguments, by the definition of $v(D)$, we have $d' \in v(D)$, which means $v(A) \subseteq v(B \multimap^{\mathsf{R}} (C \multimap^{\mathsf{R}} D))$. By the assumption, there exists X such that

1. $v(A) \subseteq v(X \multimap^{\mathsf{R}} D)$, which means that for any $a' \in v(A)$, given any $x' \in X$, $d' \in W$, if $\mathbb{R}a'x'd'$, then $d' \in v(D)$, and
2. $v(B) \subseteq v(C \multimap^{\mathsf{R}} X)$, which means that for any $b' \in v(B)$, given any $c' \in v(C)$ and $x' \in W$, if $\mathbb{R}b'c'x'$, then $x' \in v(X)$.

By $\mathbb{R}bcx$, and (2), we know that $x \in v(X)$. By $\mathbb{R}axd$, and (1), we know that $d \in v(D)$, which means that there exists $y \in W$ such that $\mathbb{R}aby$ and $\mathbb{R}ycd$.

The other cases are similar to the arguments above. □

A frame $\langle W, \leq, \mathbb{I}, \mathbb{L}\rangle$ is left (right) skew associative if $\mathbb{L}$ satisfies LSA (RSA). For other conditions, the naming is similar. If $\langle W, \leq, \mathbb{I}, \mathbb{L}\rangle$ satisfies LSA, LSLU, and LSRU (respectively RSA, RSLU, RSRU), then it is a left (respectively right) skew frame.

We can think of a `SkMBiCA` frame $\langle W, \leq, \mathbb{I}, \mathbb{L}, \mathbb{R}\rangle$ as a combination of two ternary frames $\langle W, \leq, \mathbb{I}, \mathbb{L}\rangle$ (left skew frame) and $\langle W, \leq, \mathbb{I}, \mathbb{R}\rangle$ (right skew frame) sharing the same set of possible worlds, where the ternary relations are inter-definable by $\mathbb{L}\mathbb{R}$-reverse. Whenever $\mathbb{L}\mathbb{R}$-reverse holds, then $\langle W, \leq, \mathbb{I}, \mathbb{L}\rangle$ is left skew if and only if $\langle W, \leq, \mathbb{I}, \mathbb{R}\rangle$ is right skew. In fact, we have:

$\langle W, \leq, \mathbb{I}, \mathbb{L}\rangle$ left skew associative	$\longleftrightarrow$	$\langle W, \leq, \mathbb{I}, \mathbb{R}\rangle$ right skew associative
$\langle W, \leq, \mathbb{I}, \mathbb{L}\rangle$ left skew left unital	$\longleftrightarrow$	$\langle W, \leq, \mathbb{I}, \mathbb{R}\rangle$ right skew right unital
$\langle W, \leq, \mathbb{I}, \mathbb{L}\rangle$ left skew right unital	$\longleftrightarrow$	$\langle W, \leq, \mathbb{I}, \mathbb{R}\rangle$ right skew left unital

If we state the structural laws semantically rather than syntactically, as in the sequent calculus `SkMBiCA`, we can reformulate Theorem 5.7 without referring to sequents and valuations. For example, we can define $\otimes^{\mathsf{L}}$ on downwards closed sets of worlds as $A \otimes^{\mathsf{L}} B = \{c : \exists a \in A \ \&\ b \in B \ \&\ \mathbb{L}abc\}$ and express α as $(A \otimes^{\mathsf{L}} B) \otimes^{\mathsf{L}} C \subseteq A \otimes^{\mathsf{L}} (B \otimes^{\mathsf{L}} C)$. It is the case that α holds in a frame if and only if it satisfies LSA.

We construct a thin SkMBiC from the frame $\langle W, \leq, \mathsf{I}, \mathbb{L}, \mathbb{R}\rangle$ and provide algebraic proofs for the main theorems in [27]. The objects in the category are downwards closed subsets of W and for A, B, we have a map $A \to B$ if and only if $A \subseteq B$.

Corollary 5.8. *The category $(\mathcal{P}_{\downarrow}(W), \subseteq)$ generated from any* `SkMBiCA` *frame is a thin* SkMBiC*.*

A frame $\langle W, \leq, \mathbb{I}, \mathbb{L}\rangle$ is associative normal if it satisfies LSA and RSA simultaneously, and left (right) unital normal if LSLU and RSLU (LSRU and RSRU) are satisfied. Therefore, by Theorem 5.7, we have a thin version of the main results in [27].

Corollary 5.9. *Given any frame, for the category $(\mathcal{P}_{\downarrow}(W), \subseteq)$ generated from the frame we have:*

$(\mathbb{I}, \otimes^{\mathsf{L}})$ *left skew monoidal*	$\longleftrightarrow$	$(\mathbb{I}, \multimap^{\mathsf{L}})$ *left skew closed*
$(\mathbb{I}, \otimes^{\mathsf{R}})$ *right skew monoidal*	$\longleftrightarrow$	$(\mathbb{I}, \multimap^{\mathsf{R}})$ *right skew closed*

Moreover, if the frame satisfies $\mathbb{LR}$*-reverse then:*

$$\begin{array}{lcl}
(\mathbb{I}, \otimes^{\mathsf{L}}) \textit{ left skew monoidal} & \longleftrightarrow & (\mathbb{I}, \otimes^{\mathsf{R}}) \textit{ right skew monoidal} \\
(\mathbb{I}, \multimap^{\mathsf{L}}) \textit{ left skew closed} & \longleftrightarrow & (\mathbb{I}, \multimap^{\mathsf{R}}) \textit{ right skew closed} \\
(\mathbb{I}, \otimes^{\mathsf{L}}) \textit{ associative normal} & \longleftrightarrow & (\mathbb{I}, \otimes^{\mathsf{R}}) \textit{ associative normal} \\
(\mathbb{I}, \otimes^{\mathsf{L}}) \textit{ left unital normal} & \longleftrightarrow & (\mathbb{I}, \otimes^{\mathsf{R}}) \textit{ right unital normal} \\
(\mathbb{I}, \otimes^{\mathsf{L}}) \textit{ right unital normal} & \longleftrightarrow & (\mathbb{I}, \otimes^{\mathsf{R}}) \textit{ left unital normal} \\
(\mathbb{I}, \multimap^{\mathsf{L}}) \textit{ associative normal} & \longleftrightarrow & (\mathbb{I}, \multimap^{\mathsf{R}}) \textit{ associative normal} \\
(\mathbb{I}, \multimap^{\mathsf{L}}) \textit{ left unital normal} & \longleftrightarrow & (\mathbb{I}, \multimap^{\mathsf{R}}) \textit{ right unital normal} \\
(\mathbb{I}, \multimap^{\mathsf{L}}) \textit{ right unital normal} & \longleftrightarrow & (\mathbb{I}, \multimap^{\mathsf{R}}) \textit{ left unital normal}
\end{array}$$

6 `SkMBiCA` with Symmetry

An exchange rule can be added to both associative and non-associative Lambek calculus to allow permutation of formulae in context [22]. It is well-known that two implications $\diagdown$ and $\diagup$ collapse into one in commutative Lambek calculus, i.e. for any formulae A and B, $A \diagdown B$ is logically equivalent to $B \diagup A$. In particular, consider an axiomatic presentation of non-associative Lambek calculus with exchange $\mathsf{ex} : A \otimes B \vdash_{\mathsf{L}} B \otimes A$, both $A \diagdown B \vdash_{\mathsf{L}} B \diagup A$ and $B \diagup A \vdash_{\mathsf{L}} A \diagdown B$ are provable. We adapt the notations in [22, Section 4] to fit in our discussion.

$$\dfrac{\dfrac{\dfrac{}{(A \diagdown B) \otimes A \vdash_{\mathsf{L}} A \otimes (A \diagdown B)}\ \mathsf{ex} \quad \dfrac{\dfrac{}{A \diagdown B \vdash_{\mathsf{L}} A \diagdown B}\ \mathsf{id}}{A \otimes (A \diagdown B) \vdash_{\mathsf{L}} B}\ \pi_{\diagdown}^{-1}}{(A \diagdown B) \otimes A \vdash_{\mathsf{L}} B}\ \mathsf{comp}}{A \diagdown B \vdash_{\mathsf{L}} B \diagup A}\ \pi_{\diagup}$$

$$\dfrac{\dfrac{\dfrac{}{A \otimes (B \diagup A) \vdash_{\mathsf{L}} (B \diagup A) \otimes A}\ \mathsf{ex} \quad \dfrac{\dfrac{}{(B \diagup A) \vdash_{\mathsf{L}} B \diagup A}\ \mathsf{id}}{(B \diagup A) \otimes A \vdash_{\mathsf{L}} B}\ \pi_{\diagup}^{-1}}{A \otimes (B \diagup A) \vdash_{\mathsf{L}} B}\ \mathsf{comp}}{B \diagup A \vdash_{\mathsf{L}} A \diagdown B}\ \pi_{\diagdown}$$

It leads to a natural question to consider semi-substructural logics with permutation. An immediate idea is to add the the following axiom to `LSkA`:

$$\dfrac{}{A \otimes B \vdash_{\mathsf{L}} B \otimes A}\ \mathsf{ex}$$

Following this axiom, we can define a derivable rule ex' that swaps any two adjacent formulae in the antecedent. This rule is defined through combinations of the axioms ex and id and the rules comp and $\otimes$. For example, given a derivation $f : (A \otimes B) \otimes C \vdash_{\mathsf{L}} D$ and the goal sequent $(B \otimes A) \otimes C \vdash_{\mathsf{L}} D$, we can use the derivable rule:

$$\dfrac{\overset{f}{(A \otimes B) \otimes C \vdash_{\mathsf{L}} D}}{(B \otimes A) \otimes C \vdash_{\mathsf{L}} D}\ \mathsf{ex}'$$

$$= \dfrac{\dfrac{\dfrac{}{B \otimes A \vdash_{\mathsf{L}} A \otimes B}\ \mathsf{ex} \quad \dfrac{}{C \vdash_{\mathsf{L}} C}\ \mathsf{id}}{(B \otimes A) \otimes C \vdash_{\mathsf{L}} (A \otimes B) \otimes C}\ \otimes \quad \overset{f}{(A \otimes B) \otimes C \vdash_{\mathsf{L}} D}}{(B \otimes A) \otimes C \vdash_{\mathsf{L}} D}\ \mathsf{comp}$$

However, as observed by Bourke and Lack [7], the axiom ex makes the calculus fully normal, i.e. λ^{-1}, ρ^{-1}, and α^{-1} are provable.

$$\lambda^{-1} \quad = \quad \dfrac{\dfrac{}{A \otimes \mathsf{I} \vdash_{\mathsf{L}} \mathsf{I} \otimes A}\ \mathsf{ex} \qquad \dfrac{}{\mathsf{I} \otimes A \vdash_{\mathsf{L}} A}\ \lambda}{A \otimes \mathsf{I} \vdash_{\mathsf{L}} A}\ \mathsf{comp}$$

$$\rho^{-1} \quad = \quad \dfrac{\dfrac{}{A \vdash_{\mathsf{L}} A \otimes \mathsf{I}}\ \rho \qquad \dfrac{}{A \otimes \mathsf{I} \vdash_{\mathsf{L}} \mathsf{I} \otimes A}\ \mathsf{ex}}{A \vdash_{\mathsf{L}} \mathsf{I} \otimes A}\ \mathsf{comp}$$

$$\alpha^{-1} \quad = \quad \dfrac{\dfrac{\dfrac{\dfrac{}{(C \otimes B) \otimes A \vdash_{\mathsf{L}} C \otimes (B \otimes A)}\ \alpha \qquad \dfrac{\dfrac{\dfrac{}{(A \otimes B) \otimes C \vdash_{\mathsf{L}} (A \otimes B) \otimes C}\ \mathsf{id}}{(B \otimes A) \otimes C \vdash_{\mathsf{L}} (A \otimes B) \otimes C}\ \mathsf{ex}'}{C \otimes (B \otimes A) \vdash_{\mathsf{L}} (A \otimes B) \otimes C}\ \mathsf{ex}'}{(C \otimes B) \otimes A \vdash_{\mathsf{L}} (A \otimes B) \otimes C}\ \mathsf{comp}}{(B \otimes C) \otimes A \vdash_{\mathsf{L}} (A \otimes B) \otimes C}\ \mathsf{ex}'}{A \otimes (B \otimes C) \vdash_{\mathsf{L}} (A \otimes B) \otimes C}\ \mathsf{ex}'$$

Therefore semi-substructural logics need a different treatment of commutativity.

Veltri has recently investigated the proof theory of *symmetric* left skew monoidal categories and *symmetric* left skew closed categories [31, 34]. These are variants of Mac Lane's symmetric monoidal categories and de Shippers' symmetric closed categories [11] which are originally introduced by Bourke and Lack [7] where the natural isomorphism representing symmetry involves *three* objects rather than two. Following the design of axiomatic calculus (called Hilbert-style calculus in the original papers) in Veltri's studies, where symmetry is represented by the following axioms (notations are modified to fit our discussion):

$$\dfrac{}{(A \otimes B) \otimes C \vdash_{\mathsf{L}} (A \otimes C) \otimes B}\ s \qquad \dfrac{}{B \multimap (A \multimap C) \vdash_{\mathsf{L}} A \multimap (B \multimap C)}\ s'$$

The axiom s is introduced for the axiomatic calculus of symmetric left skew monoidal categories where $\multimap$ is not present, while s' is the dual case for symmetric left skew closed categories.

These axioms only take care of symmetric left skew categories. In the remainder of the section, we first extend the proof-theoretical analysis to symmetric right skew and symmetric skew monidal bi-closed categories. We will first introduce the definition of symmetric left (and righta) skew monoidal closed categories then prove the equivalence of the axioms of symmetry proof-theoretically. After that we introduce the commutative extension of `SkMBiCA` (`SkMBiCT`), called $\texttt{SkMBiCA}_{\mathsf{e}}$ ($\texttt{SkMBiCT}_{\mathsf{e}}$) and prove the equivalence of the axiomatic and tree calculi. Finally, we prove that $\texttt{SkMBiCA}_{\mathsf{e}}$ is sound and complete with respect to the preordered ternary relation model and extend the correspondence theorem 5.7 with axioms of symmetry.

Definition 6.1. *A* symmetric left skew monoidal closed category $\mathbb{C}$ *is a left skew monoidal closed category equipped with a natural isomorphism* $s_{A,B,C} : (A \otimes B) \otimes C \to (A \otimes C) \otimes B$ *satisfying the equations in Figure 2.*

Similar to left skew monoidal closed categories, left skew symmetric monoidal closed categories admit an equivalent characterization, i.e. the natural isomorphism s is bijective with the natural isomorphism $s' : B \multimap (A \multimap C) \to A \multimap$

$$
\begin{array}{ccccc}
((A \otimes B) \otimes C) \otimes D & \xrightarrow{s_{A \otimes B, C, D}} & ((A \otimes B) \otimes D) \otimes C & \xrightarrow{s_{A,B,D} \otimes C} & ((A \otimes D) \otimes B) \otimes C \\
{\scriptstyle s_{A,B,C} \otimes D} \downarrow & & & & \downarrow {\scriptstyle s_{A \otimes D, B, C}} \\
((A \otimes C) \otimes B) \otimes D & \xrightarrow[s_{A \otimes C, B, D}]{} & ((A \otimes C) \otimes D) \otimes B & \xrightarrow[s_{A,C,D} \otimes B]{} & ((A \otimes D) \otimes C) \otimes B
\end{array}
$$

$$
\begin{array}{ccccc}
((A \otimes B) \otimes C) \otimes D & \xrightarrow{s_{A,B,C} \otimes D} & ((A \otimes C) \otimes B) \otimes D & \xrightarrow{s_{A \otimes C, B, D}} & ((A \otimes C) \otimes D) \otimes B \\
{\scriptstyle \alpha_{A \otimes B, C, D}} \downarrow & & & & \downarrow {\scriptstyle \alpha_{A,C,D} \otimes B} \\
(A \otimes B) \otimes (C \otimes D) & & \xrightarrow[s_{A,B,C \otimes D}]{\qquad\qquad\qquad} & & (A \otimes (C \otimes D)) \otimes B
\end{array}
$$

$$
\begin{array}{ccccc}
((A \otimes B) \otimes C) \otimes D & \xrightarrow{s_{A \otimes B, C, D}} & ((A \otimes B) \otimes D) \otimes C & \xrightarrow{s_{A,B,D} \otimes C} & ((A \otimes D) \otimes B) \otimes C \\
{\scriptstyle \alpha_{A,B,C} \otimes D} \downarrow & & & & \downarrow {\scriptstyle \alpha_{A \otimes D, B, C}} \\
(A \otimes (B \otimes C)) \otimes D & & \xrightarrow[s_{A, B \otimes C, D}]{\qquad\qquad\qquad} & & (A \otimes D) \otimes (B \otimes C)
\end{array}
$$

$$
\begin{array}{ccccc}
((A \otimes B) \otimes C) \otimes D & \xrightarrow{\alpha_{A,B,C} \otimes D} & (A \otimes (B \otimes C)) \otimes D & \xrightarrow{\alpha_{A, B \otimes C, D}} & A \otimes ((B \otimes C) \otimes D) \\
{\scriptstyle s_{A \otimes B, C, D}} \downarrow & & & & \downarrow {\scriptstyle A \otimes s_{B,C,D}} \\
((A \otimes B) \otimes D) \otimes C & \xrightarrow[\alpha_{A,B,D} \otimes C]{} & (A \otimes (B \otimes D)) \otimes C & \xrightarrow[\alpha_{A, B \otimes D, C}]{} & A \otimes ((B \otimes D) \otimes C)
\end{array}
$$

$$
\begin{array}{ccccc}
 & \nearrow^{s_{A,B,C}} & (A \otimes C) \otimes B & \searrow^{s_{A,C,B}} & \\
(A \otimes B) \otimes C & & = = = = = & & (A \otimes B) \otimes C
\end{array}
$$

Figure 2: Equations of morphisms in symmetric left skew monoidal closed category.

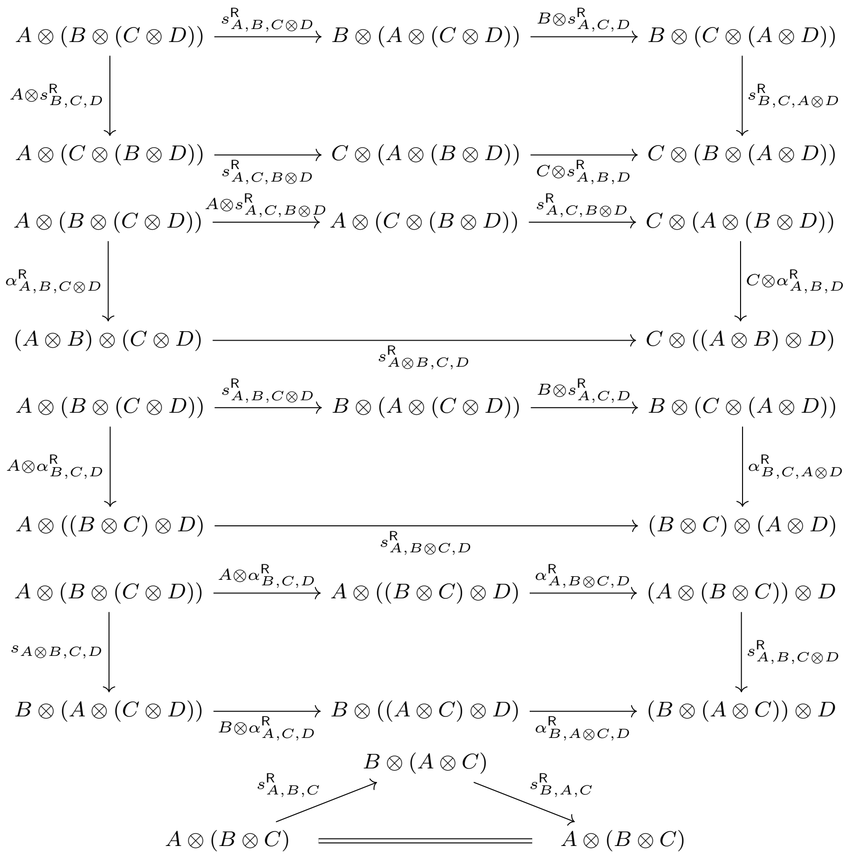


Figure 3: Equations of morphisms in symmetric right skew monoidal closed category.

$(B \multimap C)$ [7]. In other words, s' correctly characterizes symmetry in a symmetric left skew *non-monoidal* closed category.

Definition 6.2. *A* symmetric right skew monoidal closed category $\mathbb{C}$ *is a right skew monoidal closed category equipped with a natural isomorphism* $\mathsf{s}^{\mathsf{R}}_{A,B,C} : A \otimes (B \otimes C) \to B \otimes (A \otimes C)$ *satisfying the equations in Figure 3, which are similar to the ones in Figure 2 with modified bracketing.*

There exists a bijective correspondence with natural isomorphisms $\mathsf{s}'^{\mathsf{R}} : \int^{Y} Y.\mathbb{C}(B, Y \multimap D) \times \mathbb{C}(A, C \multimap Y) \to \int^{X} X.\mathbb{C}(A, X \multimap D) \times \mathbb{C}(B, C \multimap X)$ in a symmetric right skew *non-monoidal* closed category. We prove the bijective correspondence between s and s^{R} and s' and s'^{R} proof-theoretically.

Theorem 6.3. *In an axiomatic calculus of a semi-substructural logic where the adjunction of* $\otimes$ *and* $\multimap$ *are defined in the manner of Definition 2.3, if*

$$\overline{(A \otimes B) \otimes C \vdash_{\mathsf{L}} (A \otimes C) \otimes B}\ s$$

is an axiom in the calculus, then s' *is derivable and vice versa.*

Proof. <u>From s to s'.</u>

$$\dfrac{\dfrac{}{((B \multimap (A \multimap C)) \otimes A) \otimes B \vdash_{\mathsf{L}} ((B \multimap (A \multimap C)) \otimes B) \otimes A}\,\mathsf{s} \qquad \dfrac{\dfrac{\dfrac{}{B \multimap (A \multimap C) \vdash_{\mathsf{L}} B \multimap (A \multimap C)}\,\mathsf{ax}}{B \multimap (A \multimap C) \otimes B \vdash_{\mathsf{L}} A \multimap C}\,\pi^{-1}}{((B \multimap (A \multimap C)) \otimes B) \otimes A \vdash_{\mathsf{L}} C}\,\pi^{-1}}{\dfrac{((B \multimap (A \multimap C)) \otimes A) \otimes B \vdash_{\mathsf{L}} C}{\dfrac{(B \multimap (A \multimap C)) \otimes A \vdash_{\mathsf{L}} B \multimap C}{B \multimap (A \multimap C) \vdash_{\mathsf{L}} A \multimap (B \multimap C)}\,\pi}\,\pi}\,\mathsf{comp}$$

<u>From s' to s.</u>

$$\dfrac{\dfrac{\dfrac{\dfrac{}{(A \otimes C) \otimes B \vdash_{\mathsf{L}} (A \otimes C) \otimes B}\,\mathsf{ax}}{A \otimes C \vdash_{\mathsf{L}} B \multimap ((A \otimes C) \otimes B)}\,\pi}{A \vdash_{\mathsf{L}} C \multimap (B \multimap ((A \otimes C) \otimes B))}\,\pi \qquad \dfrac{}{C \multimap (B \multimap ((A \otimes C) \otimes B)) \vdash_{\mathsf{L}} B \multimap (C \multimap ((A \otimes C) \otimes B))}\,\mathsf{s}'}{\dfrac{A \vdash_{\mathsf{L}} B \multimap (C \multimap ((A \otimes C) \otimes B))}{\dfrac{A \otimes B \vdash_{\mathsf{L}} C \multimap ((A \otimes C) \otimes B)}{(A \otimes B) \otimes C \vdash_{\mathsf{L}} (A \otimes C) \otimes B}\,\pi^{-1}}\,\pi^{-1}}\,\mathsf{comp}$$

□

Theorem 6.4. *In an axiomatic calculus of a semi-substructural logic where the adjunction of $\otimes$ and $\multimap$ are defined in the manner of Definition 2.3, if*

$$\dfrac{}{A \otimes (B \otimes C) \vdash_{\mathsf{L}} B \otimes (A \otimes C)}\, s^{\mathsf{R}}$$

is an axiom then the statement

s'^{R} : *If there exists a formula Y such that two sequents $B \vdash_{\mathsf{L}} Y \multimap D$ and $A \vdash_{\mathsf{L}} C \multimap Y$ hold, then there exists a formula X such that two sequents $A \vdash_{\mathsf{L}} X \multimap D$ and $B \vdash_{\mathsf{L}} C \multimap X$ hold.*

is true.
Conversely, if s'^{R} is true in the calculus, then s^{R} is derivable.
In this context, we overload the notations X and Y to represent unknown formulae rather than atomic ones.

Proof. <u>From s^{R} to s'^{R}.</u> Suppose that there exists a formula Y such that two sequents $B \vdash_{\mathsf{L}} Y \multimap D$ and $A \vdash_{\mathsf{L}} C \multimap Y$ hold, then we take $X = B \otimes C$ and construct the desired sequents $A \vdash_{\mathsf{L}} (B \otimes C) \multimap D$ and $B \vdash_{\mathsf{L}} C \multimap (B \otimes C)$ as follows:

$$\dfrac{\dfrac{}{A \otimes (B \otimes C) \vdash_{\mathsf{L}} B \otimes (A \otimes C)}\, s^{\mathsf{R}} \qquad \dfrac{\dfrac{\dfrac{}{B \vdash_{\mathsf{L}} B}\,\mathsf{id} \quad \dfrac{\overset{\text{Assumption}}{A \vdash_{\mathsf{L}} C \multimap Y}}{A \otimes C \vdash_{\mathsf{L}} Y}\,\pi^{-1}}{B \otimes (A \otimes C) \vdash_{\mathsf{L}} B \otimes Y}\,\otimes \qquad \dfrac{\overset{\text{Assumption}}{B \vdash_{\mathsf{L}} Y \multimap D}}{B \otimes Y \vdash_{\mathsf{L}} D}\,\pi^{-1}}{B \otimes (A \otimes C) \vdash_{\mathsf{L}} D}\,\mathsf{comp}}{\dfrac{A \otimes (B \otimes C) \vdash_{\mathsf{L}} D}{A \vdash_{\mathsf{L}} (B \otimes C) \multimap D}\,\pi}\,\mathsf{comp}$$

$$\dfrac{\dfrac{}{B \otimes C \vdash_{\mathsf{L}} B \otimes C}\,\mathsf{id}}{B \vdash_{\mathsf{L}} C \multimap (B \otimes C)}\,\pi$$

Then the formula X is $B \otimes C$, where $B \vdash_{\mathsf{L}} C \multimap (B \otimes C)$ is derivable.
<u>From s'^{R} to s'^{R}.</u> To prove the sequent $A \otimes (B \otimes C) \vdash_{\mathsf{L}} B \otimes (A \otimes C)$, we start from the following two axiom sequents id : $B \otimes (A \otimes C) \vdash_{\mathsf{L}} B \otimes (A \otimes C)$ and

$\mathsf{id} : A \otimes C \vdash_{\mathsf{L}} A \otimes C$. By applying π on both sequents, we obtain $\pi\ \mathsf{id} : B \vdash_{\mathsf{L}} (A \otimes C) \multimap (B \otimes (A \otimes C))$ and $\pi\ \mathsf{id} : A \vdash_{\mathsf{L}} C \multimap (A \otimes C)$. We take $A \otimes C = Y$ to apply s'^{R}, then there exists a formula X such that two sequents $A \vdash_{\mathsf{L}} X \multimap (B \otimes (A \otimes C))$ and $B \vdash_{\mathsf{L}} C \multimap X$ hold. The desired derivation is constructed as follows:

$$\dfrac{\dfrac{\dfrac{}{A \vdash_{\mathsf{L}} A}\ \mathsf{id} \quad \dfrac{\overset{\text{By } s'^{\mathsf{R}}}{B \vdash_{\mathsf{L}} C \multimap X}}{B \otimes C \vdash_{\mathsf{L}} X}\ \pi^{-1}}{A \otimes (B \otimes C) \vdash_{\mathsf{L}} A \otimes X}\ \otimes \quad \dfrac{\overset{\text{By } s'^{\mathsf{R}}}{A \vdash_{\mathsf{L}} X \multimap (B \otimes (A \otimes C))}}{A \otimes X \vdash_{\mathsf{L}} B \otimes (A \otimes C)}\ \pi^{-1}}{A \otimes (B \otimes C) \vdash_{\mathsf{L}} B \otimes (A \otimes C)}\ \mathsf{comp}$$

□

Definition 6.5. *A symmetric skew monoidal bi-closed category* SymSkMBiC *is a skew monoidal bi-closed category with the left skew symmetry s. s^{R} is defined as $B \otimes^{\mathsf{L}} \gamma \circ \gamma \circ s \circ \gamma^{-1} \circ A \otimes^{\mathsf{R}} \gamma^{-1}$, diagrammatically:*

$$\begin{array}{ccccc} A \otimes^{\mathsf{R}} (B \otimes^{\mathsf{R}} C) & \xrightarrow{A \otimes^{\mathsf{R}} \gamma^{-1}} & A \otimes^{\mathsf{R}} (C \otimes^{\mathsf{L}} B) & \xrightarrow{\gamma^{-1}} & (C \otimes^{\mathsf{L}} B) \otimes^{\mathsf{L}} A \\ \downarrow s^{\mathsf{R}} & & & & \downarrow s \\ B \otimes^{\mathsf{R}} (A \otimes^{\mathsf{R}} C) & \xleftarrow[B \otimes^{\mathsf{R}} \gamma]{} & B \otimes^{\mathsf{R}} (C \otimes^{\mathsf{L}} A) & \xleftarrow[\gamma]{} & (C \otimes^{\mathsf{L}} A) \otimes^{\mathsf{L}} B \end{array}$$

The axiomatic calculus that is sound and complete with respect to SymSkMBiC is $\mathtt{SkMBiCA_e}$ which is extended from `SkMBiCA` by adding the axiom:

$$\dfrac{}{(A \otimes^{\mathsf{L}} B) \otimes^{\mathsf{L}} C \vdash_{\mathsf{L}} (A \otimes^{\mathsf{L}} C) \otimes^{\mathsf{L}} B}\ s$$

The axiom s^{R} is defined by transforming the diagram in Definition 6.5 into a proof in $\mathtt{SkMBiCA_e}$, and then by Theorems 6.3 and 6.4, s' and s'^{R} are derivable in $\mathtt{SkMBiCA_e}$.

Moreover, we can construct the free SymSkMBiC (FSymSkMBiC(At)) over a set At by a similar construction of FSkMBiC(At) in Section 4:

- Objects of FSymSkMBiC(At) are formulae (Fma).
- Morphisms between A and B are derivations of sequents $A \vdash_{\mathsf{L}} B$ and identified up to the congruence relation $\doteq$ defined in Figure 1 with following additional equations:

$$\begin{array}{ll} \text{(sym. axioms)} & \begin{array}{c} s \otimes^{\mathsf{L}} \mathsf{id} \circ s \circ s \otimes^{\mathsf{L}} \mathsf{id} \doteq s \circ s \otimes^{\mathsf{L}} \mathsf{id} \circ s \\ s \circ \alpha \doteq \alpha \otimes^{\mathsf{L}} \mathsf{id} \circ s \circ s \otimes^{\mathsf{L}} \mathsf{id} \quad s \circ \alpha \otimes^{\mathsf{L}} \mathsf{id} \doteq \alpha \circ s \otimes^{\mathsf{L}} \mathsf{id} \circ s \\ \alpha \circ \alpha \otimes^{\mathsf{L}} \mathsf{id} \circ s \doteq \mathsf{id} \otimes^{\mathsf{L}} s \circ \alpha \circ \alpha \otimes^{\mathsf{L}} \mathsf{id} \end{array} \\ (s \text{ symmetry}) & s \circ s \doteq \mathsf{id} \end{array}$$

On the other hand, the commutative extension of `SkMBiCT` ($\mathtt{SkMBiCT_e}$) is defined by adding the following two rules:

$$\dfrac{T[(U_0, U_1), U_2] \vdash_{\mathsf{T}} C}{T[(U_0, U_2), U_1] \vdash_{\mathsf{T}} C}\ \mathsf{ex}^{\mathsf{L}} \quad \dfrac{T[U_0; (U_1; U_2)] \vdash_{\mathsf{T}} C}{T[U_1; (U_0; U_2)] \vdash_{\mathsf{T}} C}\ \mathsf{ex}^{\mathsf{R}}$$

A result similar to Theorems 6.3 and 6.4 can also be proved in $\mathtt{SkMBiCT_e}$. We adopt a symmetric presentation to emphasize that $\mathtt{SkMBiCT_e}$ should be viewed as a combination of two distinct calculi, connected through the rule $\otimes\mathsf{comm}$.

Moreover, $\mathtt{SkMBiCA_e}$ and $\mathtt{SkMBiCT_e}$ are equivalent.

Theorem 6.6. $\mathtt{SkMBiCA_e}$ *is equivalent to* $\mathtt{SkMBiCT_e}$*, meaning that the following two statements are true:*

- *For any derivation* $f : A \vdash_{\mathsf{L}} C$*, there exists a derivation* $\mathsf{A2T}f : A \vdash_{\mathsf{T}} C$*.*
- *For any derivation* $f : T \vdash_{\mathsf{T}} C$*, there exists a derivation* $\mathsf{T2A}f : T^{\#} \vdash_{\mathsf{L}} C$*, where* $T^{\#}$ *transforms a tree into a formula by replacing commas with* $\otimes^{\mathsf{L}}$ *and semicolons with* $\otimes^{\mathsf{R}}$*, and* $-$ *with* I*, respectively.*

Proof. We extend the proof of Theorem 4.5 by examining the additional cases of s (for $\mathsf{A2T}$) and ex^{L} and ex^{R} (for $\mathsf{T2A}$).

<u>Case $f = s$</u>

$$\dfrac{}{(A \otimes^{\mathsf{L}} B) \otimes^{\mathsf{L}} C \vdash_{\mathsf{L}} (A \otimes^{\mathsf{L}} C) \otimes^{\mathsf{L}} B}\ s$$

$$\mapsto \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash_{\mathsf{T}} A}\ \mathsf{ax} \quad \dfrac{}{C \vdash_{\mathsf{T}} C}\ \mathsf{ax}}{A, C \vdash_{\mathsf{T}} A \otimes^{\mathsf{L}} C}\ \otimes^{\mathsf{L}}\mathsf{R} \quad \dfrac{}{B \vdash_{\mathsf{T}} B}\ \mathsf{ax}}{(A, C), B \vdash_{\mathsf{T}} (A \otimes^{\mathsf{L}} C) \otimes^{\mathsf{L}} B}\ \otimes^{\mathsf{L}}\mathsf{R}}{(A, B), C \vdash_{\mathsf{T}} (A \otimes^{\mathsf{L}} C) \otimes^{\mathsf{L}} B}\ \mathsf{ex}^{\mathsf{L}}}{(A \otimes^{\mathsf{L}} B), C \vdash_{\mathsf{T}} (A \otimes^{\mathsf{L}} C) \otimes^{\mathsf{L}} B}\ \otimes^{\mathsf{L}}\mathsf{L}}{(A \otimes^{\mathsf{L}} B) \otimes^{\mathsf{L}} C \vdash_{\mathsf{T}} (A \otimes^{\mathsf{L}} C) \otimes^{\mathsf{L}} B}\ \otimes^{\mathsf{L}}\mathsf{L}$$

<u>Case $f = \mathsf{ex}^{\mathsf{L}}\ f'$</u>

$$\dfrac{\overset{f'}{T[(U_0, U_1), U_2] \vdash_{\mathsf{T}} C}}{T[(U_0, U_2), U_1] \vdash_{\mathsf{T}} C}\ \mathsf{ex}^{\mathsf{L}}$$

$$\mapsto \quad \dfrac{\dfrac{\dfrac{\dfrac{}{(U_0^{\#} \otimes^{\mathsf{L}} U_2^{\#}) \otimes^{\mathsf{L}} U_1^{\#} \vdash_{\mathsf{T}} (U_0^{\#} \otimes^{\mathsf{L}} U_1^{\#}) \otimes^{\mathsf{L}} U_2^{\#}}\ s}{T[(U_0^{\#} \otimes^{\mathsf{L}} U_2^{\#}) \otimes^{\mathsf{L}} U_1^{\#}]^{\#} \vdash_{\mathsf{T}} T[(U_0^{\#} \otimes^{\mathsf{L}} U_1^{\#}) \otimes^{\mathsf{L}} U_2^{\#}]^{\#}}\ \text{Lemma 4.4}}{T[(U_0, U_2), U_1]^{\#} \vdash_{\mathsf{L}} T[(U_0, U_1), U_2]^{\#}}\ \text{Lemma 4.3} \quad \overset{\mathsf{T2A}f'}{T[(U_0, U_1), U_2]^{\#} \vdash_{\mathsf{T}} C}}{T[(U_0, U_2), U_1]^{\#} \vdash_{\mathsf{T}} C}\ \mathsf{comp}$$

<u>Case $f = \mathsf{ex}^{\mathsf{R}}\ f'$</u>

$$\dfrac{\overset{f'}{T[U_0; (U_1; U_2)] \vdash_{\mathsf{T}} C}}{T[U_1; (U_0; U_2)] \vdash_{\mathsf{T}} C}\ \mathsf{ex}^{\mathsf{R}}$$

$$\mapsto \quad \dfrac{\dfrac{\dfrac{\dfrac{}{U_1^{\#} (\otimes^{\mathsf{R}} U_0^{\#} \otimes^{\mathsf{R}} U_2^{\#}) \vdash_{\mathsf{T}} U_0^{\#} \otimes^{\mathsf{R}} (U_1^{\#} \otimes^{\mathsf{R}} U_2^{\#})}\ s^{\mathsf{R}}}{T[U_1^{\#} \otimes^{\mathsf{R}} (U_0^{\#} \otimes^{\mathsf{R}} U_2^{\#})]^{\#} \vdash_{\mathsf{T}} T[U_0^{\#} \otimes^{\mathsf{R}} (U_1^{\#} \otimes^{\mathsf{R}} U_2^{\#})]^{\#}}\ \text{Lemma 4.4}}{T[U_1; (U_0; U_2)]^{\#} \vdash_{\mathsf{L}} T[U_0; (U_1; U_2)]^{\#}}\ \text{Lemma 4.3} \quad \overset{\mathsf{T2A}f'}{T[U_0; (U_1; U_2)]^{\#} \vdash_{\mathsf{T}} C}}{T[U_1; (U_0; U_2)]^{\#} \vdash_{\mathsf{T}} C}\ \mathsf{comp}$$

□

Recall that in commutative Lambek calculus (both associative and non-associative), the two implications collapse into one. However, this is not the case in either $\mathtt{SkMBiCA_e}$ or $\mathtt{SkMBiCT_e}$. Specifically, for any formulae A and B, neither of the sequents $A \multimap^{\mathsf{L}} B \vdash_i A \multimap^{\mathsf{R}} B$ nor $A \multimap^{\mathsf{R}} B \vdash_i A \multimap^{\mathsf{L}} B$ ($i \in \{\mathsf{L}, \mathsf{T}\}$) is provable. We demonstrate this non-provability by taking A and B as atomic formulae.

$$\dfrac{\dfrac{??}{(X \multimap^{\mathsf{L}} Y) \otimes^{\mathsf{R}} X \vdash_{\mathsf{L}} Y}}{X \multimap^{\mathsf{L}} Y \vdash_{\mathsf{L}} X \multimap^{\mathsf{R}} Y}\ \pi^{\mathsf{R}} \qquad \dfrac{\dfrac{??}{(X \multimap^{\mathsf{R}} Y) \otimes^{\mathsf{L}} X \vdash_{\mathsf{L}} Y}}{X \multimap^{\mathsf{R}} Y \vdash_{\mathsf{L}} X \multimap^{\mathsf{L}} Y}\ \pi$$

$$\dfrac{\dfrac{\dfrac{??}{(X \multimap^{\mathsf{L}} Y) ; X \vdash_{\mathsf{T}} Y}}{(X \multimap^{\mathsf{L}} Y) \otimes^{\mathsf{R}} X \vdash_{\mathsf{T}} Y}\ \otimes^{\mathsf{R}}\mathsf{L}}{X \multimap^{\mathsf{L}} Y \vdash_{\mathsf{T}} X \multimap^{\mathsf{R}} Y}\ \multimap^{\mathsf{R}}\mathsf{R} \qquad \dfrac{\dfrac{\dfrac{??}{(X \multimap^{\mathsf{R}} Y) , X \vdash_{\mathsf{T}} Y}}{(X \multimap^{\mathsf{R}} Y) \otimes^{\mathsf{L}} X \vdash_{\mathsf{T}} Y}\ \otimes^{\mathsf{L}}\mathsf{L}}{X \multimap^{\mathsf{R}} Y \vdash_{\mathsf{T}} X \multimap^{\mathsf{L}} Y}\ \multimap^{\mathsf{R}}\mathsf{R}$$

Lastly, we can analyze skew symmetry through the lens of ternary relational semantics and obtain a sound and complete model of $\mathtt{SkMBiCA_e}$. Furthermore, we obtain the correspondence theorem of ternary frame conditions and validity of structural laws.

Definition 6.7. *We list the frame conditions properties of skew commutativity:*

Left Skew Commutativity (LSC)	$\forall a, b, c, d, x \in W, \mathbb{L}abx \ \&\ \mathbb{L}xcd \longrightarrow \exists y \in W$ *s.t.* $\mathbb{L}acy \ \&\ \mathbb{L}ybd$.
Right Skew Commutativity (RSC)	$\forall a, b, c, d, x \in W, \mathbb{L}bcx \ \&\ \mathbb{L}axd \longrightarrow \exists y \in W$ *s.t.* $\mathbb{L}acy \ \&\ \mathbb{L}byd$.

A $\mathtt{SkMBiCA_e}$ frame is a $\mathtt{SkMBiCA}$ frame where $\mathbb{L}$ and $\mathbb{R}$ additionally satisfy LSC and RSC, respectively. A $\mathtt{SkMBiCA_e}$ model is a $\mathtt{SkMBiCA_e}$ frame with a valuation function.

Theorem 6.8 (Soundness)**.** *If a sequent $A \vdash_{\mathsf{L}} B$ is provable in $\mathtt{SkMBiCA_e}$ then it is valid in any $\mathtt{SkMBiCA_e}$ model.*

Proof. The proof is extended from the proof of Theorem 5.3 by examining one additional case, $f = s : (A \otimes^{\mathsf{L}} B) \otimes^{\mathsf{L}} C \vdash_{\mathsf{L}} (A \otimes^{\mathsf{L}} C) \otimes^{\mathsf{L}} B$. For any $\mathtt{SkMBiCA_e}$ model $\langle W, \mathbb{I}, \mathbb{L}, \mathbb{R}, v\rangle$ and any $d \in v((A \otimes^{\mathsf{L}} B) \otimes^{\mathsf{L}} C)$, there exist $x \in v(A \otimes^{\mathsf{L}} B)$ and $c \in v(C)$ such that $\mathbb{L}xcd$. Moreover, there exist $a \in v(A)$ and $b \in v(B)$ such that $\mathbb{L}abx$. By LSC, we know that there exist $y \in W$ such that $\mathbb{L}acy$ and $\mathbb{L}ybd$, which means that $d \in v((A \otimes^{\mathsf{L}} C) \otimes^{\mathsf{L}} B)$. □

Definition 6.9. *The canonical model of* $\mathtt{SkMBiCA}$ *is* $\langle W, \leq, \mathbb{I}, \mathbb{L}, \mathbb{R}, v\rangle$ *where*

- $W = \mathsf{Fma}$ *and* $A \leq B$ *if and only if* $A \vdash_{\mathsf{L}} B$,
- $\mathbb{I} = v(\mathsf{I})$,
- $\mathbb{L}ABC$ *if and only if* $C \vdash_{\mathsf{L}} A \otimes^{\mathsf{L}} B$,
- $\mathbb{R}ABC$ *if and only if* $C \vdash_{\mathsf{L}} A \otimes^{\mathsf{R}} B$, *and*
- $v(A) = \{B \mid B \vdash_{\mathsf{L}} A$ *is provable in* $\mathtt{SkMBiCA_e}\}$.

Lemma 6.10. *The canonical model is a* $\mathtt{SkMBiCA_e}$ *model.*

Proof. The proof proceeds similarly to the proof of Lemma 5.5 but with one additional case showing that LSC is satisfied.
Given five formulae A, B, C, C', D and two derivations $f : C' \vdash_{\mathsf{L}} A \otimes^{\mathsf{L}} B$ and $g : D \vdash_{\mathsf{L}} C' \otimes^{\mathsf{L}} C$, then we take $A \otimes^{\mathsf{L}} C$ as the desired formula. The first desired sequent $A \otimes^{\mathsf{L}} C \vdash_{\mathsf{L}} A \otimes^{\mathsf{L}} C$ is derivable and the other desired sequent $D \vdash_{\mathsf{L}} (A \otimes^{\mathsf{L}} C) \otimes^{\mathsf{L}} B$ is constructed as follows:

$$\dfrac{\overset{g}{D \vdash_{\mathsf{L}} C' \otimes^{\mathsf{L}} C} \quad \dfrac{\dfrac{\overset{f}{C' \vdash_{\mathsf{L}} A \otimes^{\mathsf{L}} B} \quad \dfrac{}{C \vdash_{\mathsf{L}} C}\,\mathsf{ax}}{C' \otimes^{\mathsf{L}} C \vdash_{\mathsf{L}} (A \otimes^{\mathsf{L}} B) \otimes^{\mathsf{L}} C}\,\otimes^{\mathsf{L}} \quad \dfrac{}{(A \otimes^{\mathsf{L}} C) \otimes^{\mathsf{L}} B \vdash_{\mathsf{L}} (A \otimes^{\mathsf{L}} B) \otimes^{\mathsf{L}} C}\,s}{C' \otimes^{\mathsf{L}} C \vdash_{\mathsf{L}} (A \otimes^{\mathsf{L}} C) \otimes^{\mathsf{L}} B}\,\mathsf{comp}}{D \vdash_{\mathsf{L}} (A \otimes^{\mathsf{L}} C) \otimes^{\mathsf{L}} B}\,\mathsf{comp}$$

□

Following the same argument in the proof of Theorem 5.6, we have:

Theorem 6.11 (Completeness)**.** *If $A \vdash_{\mathsf{L}} B$ is valid in any* $\mathtt{SkMBiCA_e}$ *model, then it is provable in* $\mathtt{SkMBiCA_e}$.

Finally, we extend the correspondence between frame conditions and validity of structural laws to the symmetric case.

Theorem 6.12. *For any ternary frame $\langle W, \leq, \mathbb{I}, \mathbb{L}\rangle$,*

$$\begin{array}{ccccc} s \textit{ valid} & \longleftrightarrow & \textit{LSC holds} & \longleftrightarrow & s' \textit{ valid} \\ s^{\mathsf{R}} \textit{ valid} & \longleftrightarrow & \textit{RSC holds} & \longleftrightarrow & s'^{\mathsf{R}} \textit{ valid} \end{array}$$

Proof. s : LSC holds if and only if s is valid.

- ($\longrightarrow$) This is similar to the case of s in the proof of Theorem 6.8.
- ($\longleftarrow$) Suppose that s is valid, i.e. for any A, B, C, $v((A \otimes^{\mathsf{L}} B) \otimes^{\mathsf{L}} C) \subseteq v((A \otimes^{\mathsf{L}} C) \otimes^{\mathsf{L}} B)$. Consider any $a, b, c, d, x \in W$ such that $\mathbb{L}abx$ and $\mathbb{L}xcd$. We take $v(A) = a{\downarrow}, v(B) = b{\downarrow}$, $v(C) = c{\downarrow}$ for some $A, B, C \in \mathsf{At}$, then we know that $x \in v(A \otimes^{\mathsf{L}} B)$ and $d \in v((A \otimes^{\mathsf{L}} B) \otimes^{\mathsf{L}} C)$. By the assumption, $d \in v((A \otimes^{\mathsf{L}} C) \otimes^{\mathsf{L}} B)$ as well, which means that there exist $a', b', y, c' \in W$ such that $\mathbb{L}a'c'y$ and $\mathbb{L}yb'd$. Because $\mathbb{L}$ is upward closed in its first and second argument, we have $\mathbb{L}acy$ and $\mathbb{L}ybd$ as desired.

s' : LSC holds if and only if s' is valid.

- ($\longrightarrow$) Suppose that LSC holds, we show that for any A, B, C, $v(B \multimap^{\mathsf{L}} (A \multimap^{\mathsf{L}} C)) \subseteq v(A \multimap^{\mathsf{L}} (B \multimap^{\mathsf{L}} C))$. Consider any $d \in v(B \multimap^{\mathsf{L}} (A \multimap^{\mathsf{L}} C))$. Assume that there exists $a \in v(A)$, $b \in v(B)$, and $x, c \in W$ such that $\mathbb{L}dax$ and $\mathbb{L}xbc$. Our goal is to prove that $c \in v(C)$. By LSC, there exists $y \in W$ such that $\mathbb{L}dby$ and $\mathbb{L}yac$, then by the assumption $d \in v(B \multimap^{\mathsf{L}} (A \multimap^{\mathsf{L}} C))$, we know that $c \in v(C)$.
- ($\longleftarrow$) Suppose that s' is valid, i.e. for any A, B, C, $v(B \multimap^{\mathsf{L}} (A \multimap^{\mathsf{L}} C)) \subseteq v(A \multimap^{\mathsf{L}} (B \multimap^{\mathsf{L}} C))$. Consider any $a, b, c, d, x \in W$ such that $\mathbb{L}abx$ and $\mathbb{L}xcd$. Take $v(A) = b{\downarrow}$, $v(B) = c{\downarrow}$, and $v(C) = \{d' \mid \exists y.\mathbb{L}acy \& \mathbb{L}ybd\}$ for some $A, B, C \in \mathsf{At}$. Consider any $c' \in v(B)$,

$b' \in v(A)$, $y', d' \in W$, $\mathbb{L}ac'y'$ and $\mathbb{L}y'b'd'$. Because $\mathbb{L}$ is upwards closed in its second argument, we have $\mathbb{L}acy'$ and $\mathbb{L}y'bd'$, which means that $y' \in v(A \multimap^{\mathsf{L}} C)$ and $d' \in v(C)$, therefore $a \in v(B \multimap^{\mathsf{L}} (A \multimap^{\mathsf{L}} C))$. By validity of s', $\mathbb{L}abx$, and $\mathbb{L}xcd$, we know that $d \in v(C)$, i.e. there exists $y \in W$ such that $\mathbb{L}acy$ and $\mathbb{L}ybd$.

s^{R} : RSC holds if and only if s^{R} is valid.

($\longrightarrow$) Suppose that RSC holds, we show that for any A, B, C, $v(A \otimes^{\mathsf{R}} (B \otimes^{\mathsf{R}} C)) \subseteq v(B \otimes^{\mathsf{R}} (A \otimes^{\mathsf{R}} C))$. Consider any $d \in v(A \otimes^{\mathsf{R}} (B \otimes^{\mathsf{R}} C))$. By definition, there exists $a \in v(A)$, $b \in v(B)$, $c \in v(C)$, $x \in v(B \otimes^{\mathsf{R}} C)$ such that $\mathbb{L}bcx$ and $\mathbb{L}axd$. By RSC, there exists $y \in W$ such that $\mathbb{L}acy$ and $\mathbb{L}byd$, then by definition, we know that $y \in v(A \otimes^{\mathsf{R}} C)$ and therefore $d \in v(B \otimes^{\mathsf{R}} (A \otimes^{\mathsf{R}} C))$.

($\longleftarrow$) Suppose that s^{R} is valid. Consider any $a, b, c, d, x \in W$ such that $\mathbb{L}bcx$ and $\mathbb{L}axd$. We take $v(A) = a{\downarrow}, v(B) = b{\downarrow}$, $v(C) = c{\downarrow}$ for some $A, B, C \in \mathsf{At}$, then we know that $x \in v(B \otimes^{\mathsf{R}} C)$ and $d \in v(A \otimes^{\mathsf{R}} (B \otimes^{\mathsf{R}} C))$. By the assumption, $d \in v(B \otimes^{\mathsf{R}} (A \otimes^{\mathsf{R}} C))$ as well, which means that that there exist $a', b', y, c' \in W$ such that $\mathbb{L}a'c'y$ and $\mathbb{L}b'yd$. Because $\mathbb{L}$ is upwards closed in its first and second argument, we have $\mathbb{L}acy$ and $\mathbb{L}byd$ as desired.

s'^{R} : RSC holds if and only if s'^{R} is valid.

($\longrightarrow$) Suppose that RSC holds, we show that for any formulae A, B, C, D, if there exists a formula Y such that $v(B) \subseteq v(Y \multimap^{\mathsf{R}} D)$ and $v(A) \subseteq v(C \multimap^{\mathsf{R}} Y)$ then there exists a formula X such that $v(A) \subseteq v(X \multimap^{\mathsf{R}} D)$ and $v(B) \subseteq v(C \multimap^{\mathsf{R}} X)$. Take $X = B \otimes^{\mathsf{R}} C$, then clearly $v(B) \subseteq v(C \multimap^{\mathsf{R}} (B \otimes^{\mathsf{R}} C))$. For any $a \in v(A)$, if there exist $x \in v(B \multimap^{\mathsf{R}} C)$ and $d \in W$ such that $\mathbb{L}axd$, then by definition, there exist $b \in v(B)$ and $c \in v(C)$ such that $\mathbb{L}bcx$. By RSC, there exists $y \in W$ such that $\mathbb{L}acy$ and $\mathbb{L}byd$, then by $v(B) \subseteq v(Y \multimap^{\mathsf{R}} D)$, $d \in v(D)$, therefore $a \in v(X \multimap^{\mathsf{R}} D)$.

($\longleftarrow$) Suppose that s'^{R} is valid. Consider any $a, b, c, d, x \in W$ such that $\mathbb{L}bcx$ and $\mathbb{L}axd$. Take $v(A) = a{\downarrow}$, $v(B) = b{\downarrow}$, $v(C) = c{\downarrow}$, and $v(D) = \{d' \mid \exists y. \mathbb{L}acy \& \mathbb{L}byd\}$ for some $A, B, C, D \in \mathsf{At}$. Clearly, $v(A)$ is a subset of $v(C \multimap^{\mathsf{R}} (A \otimes^{\mathsf{R}} C))$. For any $b' \in v(B)$, if there exist $y' \in v(A \otimes^{\mathsf{R}} C)$ and $d' \in W$ and $\mathbb{L}b'y'd'$, then by definition, there exist $a' \in v(A)$ and $c' \in v(C)$ such that $\mathbb{L}a'c'y'$. Because $\mathbb{L}$ is upwards closed in its first and second argument, we have $\mathbb{L}acy'$ and $\mathbb{L}by'd'$, which means that $d' \in v(D)$ and therefore $v(B) \in v((A \otimes^{\mathsf{R}} C) \multimap^{\mathsf{R}} D)$. Take $F = A \otimes^{\mathsf{R}} C$, then by s'^{R}, there exists a formula E such that $v(A) \subseteq v(E \multimap^{\mathsf{R}} D)$ and $v(B) \subseteq v(C \multimap^{\mathsf{R}} E)$. By $b \in v(C \multimap^{\mathsf{R}} E)$ and $\mathbb{L}bcx$, we have $x \in v(E)$. By $a \in v(E \multimap^{\mathsf{R}} D)$ and $\mathbb{L}axd$, we have $d \in v(D)$, which means that there exists $y \in W$ such that $\mathbb{L}acy$ and $\mathbb{L}byd$, as desired.

$\square$

7 Concluding remarks

This paper discusses sequent calculi for (symmetric) left (right) skew monoidal categories and (symmetric) skew monoidal bi-closed categories in the style of non-associative Lambek calculus. Compared to the sequent calculi with stoup, although the calculi à la Lambek are not immediately decidable but are more flexible in the sense that the sequent calculi for right skew monoidal closed categories (`RSkT`) and skew monoidal bi-closed categories (`SkMBiCT`) are presentable. Moreover, we show that they are cut-free and equivalent to the calculus with stoup (Theorem 2.11) and the axiomatic calculus (Theorem 4.5).

Moreover, we discuss the relational semantics of `SkMBiCA` ($\mathtt{SkMBiCA_e}$) via the ternary frame $\langle W, \leq, \mathbb{I}, \mathbb{L}, \mathbb{R}\rangle$ where $\mathbb{L}$ and $\mathbb{R}$ are connected by $\mathbb{LR}$-reverse and therefore if $\mathbb{L}$ satisfies left skew structural conditions then $\mathbb{R}$ satisfies right skew structural conditions automatically. By Theorem 5.7, for any `SkMBiCA` model, we can construct a thin skew monoidal bi-closed category $(\mathcal{P}_{\downarrow}(W), \subseteq)$ and obtain algebraic proofs of main theorems in [27].

A deeper exploration of symmetric right skew closed categories remains as future work, particularly in identifying appropriate coherence conditions without relying on monoidal structures. This investigation builds upon the foundational classification of closed categories by Day and Laplaza [10], which ranges from symmetric monoidal closed through symmetric closed and closed, to non-associative closed categories. Their work provided concrete examples where the Day convolution version of structural laws fails to be bijective, but did not address the symmetric non-associative variant. In Section 6, we established results for the special case of posetal (thin) symmetric skew monoidal bi-closed categories, where there is at most one morphism between any pair of objects. The natural progression is to extend these results to non-posetal categories, requiring again the coherence conditions for symmetric right skew closed categories. This extension will extend the Eilenger-Kelly theorem [13, 27] to the symmetric skew monoidal closed categories.

Another possible future direction is to incorporate modalities (exponentials in linear logical terminology) with semi-substructural logic as in [20] (modalities) and [4] (subexponentials) with non-associative Lambek calculus and non-commutative and non-associative linear logic.

Similar to the equational theories for `SkMBiCA` discussed in Section 4, we also plan to investigate the equational theories on the derivations of `LSkT` and `SkMBiCT` in the future as well as their commutative version.

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